Lecture 16: Energy Methods

- **Announcements:**
- HW#4 online and due Monday, March 14, 5 p.m.
- Module 9 on Energy Methods online
- Midterm Exam, Thursday, March 17, 5:10-7 p.m. in 3107 Etcheverry
- Passed out old midterm solutions and went through midterm info sheet
- ---------------------------------------
- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients for folded-beam devices
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- Reading: Senturia, Chpt. 10
- Lecture Topics:
  - Energy Methods
  - Virtual Work
  - Energy Formulations
  - Tapered Beam Example
  - Estimating Resonance Frequency
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- **Last Time:**
- Deriving the differential equation governing the behavior of a tensioned beam

Tensioned Spring (Non-Ideality)

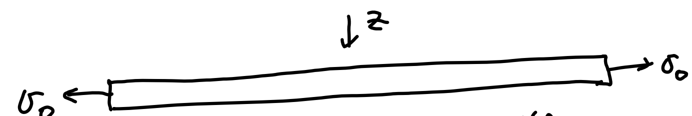
- Important case for MEMS suspensions, since the thin films comprising them are often under residual stress
- Consider small deflection case: $y(x) \ll L$

Governing differential equation: (Euler Beam Equation)

$$EI_z \frac{d^4y}{dx^4} - \underset{\text{Axial Load}}{S}\frac{d^2y}{dx^2} = \underbrace{F\delta(x-L)}_{\text{Unit impulse @ x=L}}$$

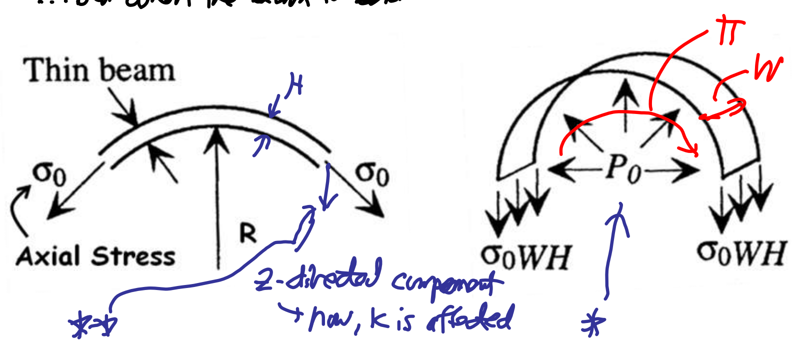
Heuristic Derivation for the Euler Beam Equation

Consider first a straight beam under axial stress:



→ no effect on z-directed stiffness when the beam is straight

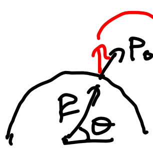
...but when the beam is bent:



* Upward pressure P_o to counteract the
** downward force from — to keep everything in static equilibrium

For ease of analysis:
Assume the beam is bent to an angle π
↳ Downward vertical force: $2\sigma_o W H$

Upward Force due to P_o:



$P_{up}(\theta) = P_o \sin\theta$

$F_u = \int_0^{\pi} (P_o \sin\theta) W (R d\theta)$

$= -P_o W R \cos\theta \Big|_0^{\pi}$

$= 2RWP_o$

[Equilibrium] ⇒ $2RWP_o = 2\sigma_o W H \rightarrow P_o = \dfrac{\sigma_o H}{R}$

$\left[q_o = \dfrac{\text{beam load}}{\text{unit length}} = P_o W, \quad \dfrac{1}{R} = \dfrac{d^2 w}{dx^2} \right]$ — beam displacement in z-direction

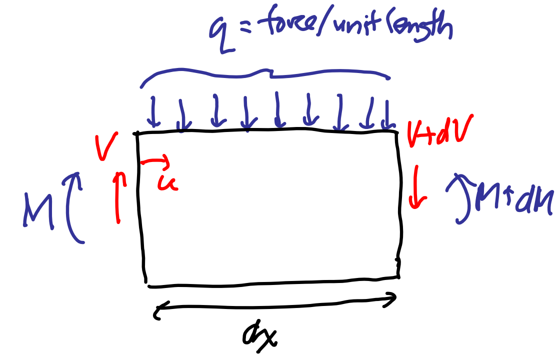
$q_o = \sigma_o W H \dfrac{d^2 w}{dx^2}$ ← generalizes to the case of small displacement angles

Using the differential beam bending equation

$\dfrac{d^2 w}{dx^2} = -\dfrac{M}{EI} \xrightarrow{???} \dfrac{d^4 w}{dx^4} = \dfrac{q}{EI}$ ← $\dfrac{\text{load}}{\text{unit length}}$

↓ *

* ↳ Relationships Between Forces on a Fully Loaded Differential Beam Element



[Total Static Equilibrium] ⇒ total force = 0

F_T = total force $= q\,dx + (V + dV) - V = 0$

$\therefore \boxed{\dfrac{dV}{dx} = -q}$ (1)

⇒ Also, total moment w/r to left hand edge = 0

$M_T = (\cancel{M} + dM) - \cancel{M} - (V + \cancel{dV})\,dx - \cancel{\tfrac{1}{2} q\,dx^2} = 0$

[neglect product of differentials]

$\int_0^{dx} (q\,du)u = \tfrac{1}{2} q\,dx^2$

$dM - V\,dx = 0 \rightarrow \boxed{\dfrac{dM}{dx} = V}$ (2)

Using (1) & (2):

$\left[\dfrac{d^2 M}{dx^2} = \dfrac{dV}{dx} = -q \right]$

$EI \dfrac{d^4 w}{dx^4} = q + q_o$ ← q: external load; q_o: equiv. load from axial stress

$$\left[q_0 = \sigma_0 W H \frac{d^2 w}{dx^2} \right]$$

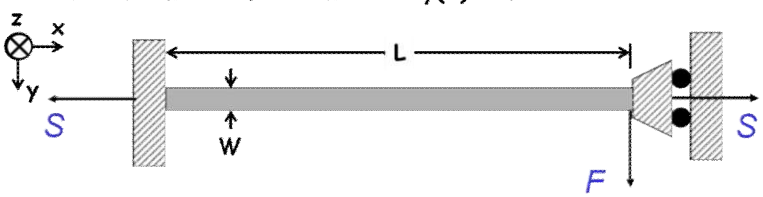
$$EI \frac{d^4 w}{dx^4} - \underbrace{(\sigma_0 W H)}_{\text{tension in the beam} = S} \frac{d^2 w}{dx^2} = q$$

a force

Euler Beam Equation

$\boxed{\text{Clamp-Guided Beam Under Axial Stress}}$

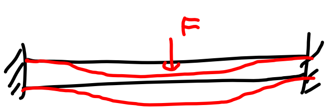
- Important case for MEMS suspensions, since the thin films comprising them are often under residual stress
- Consider small deflection case: y(x) « L



Governing differential equation: (Euler Beam Equation)

$$EI_z \frac{d^4 y}{dx^4} - S\frac{d^2 y}{dx^2} = \underbrace{F\delta(x-L)}_{\text{Unit impulse @ x=L}}$$

Axial Load



Senturia solves this...

- Can solve the ODE using standard methods
  - Senturia, pp. 232-235: solves ODE for case of point load on a clamped-clamped beam (which defines B.C.'s)
  - For solution to the clamped-guided case: see S. Timoshenko, Strength of Materials II: Advanced Theory and Problems, McGraw-Hill, New York, 3rd Ed., 1955
- Result from Timoshenko:

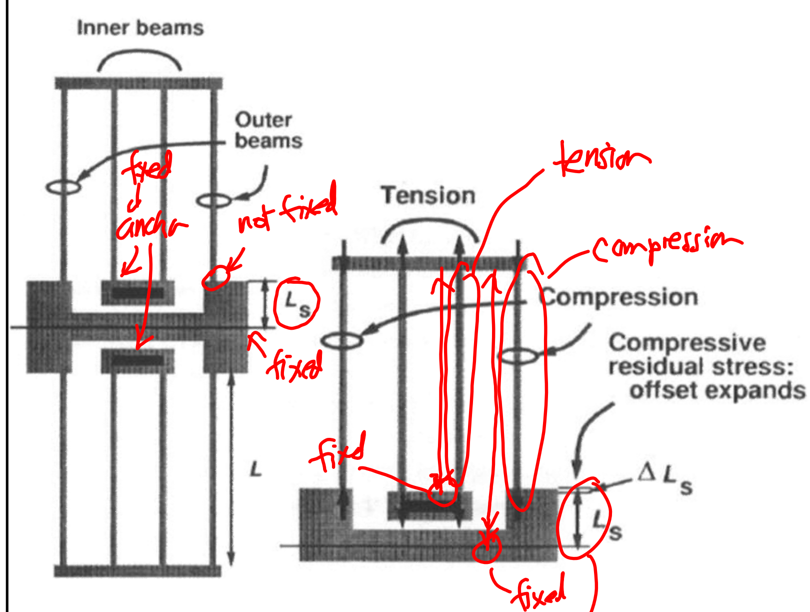
S > 0 (tension) $\rightarrow$ $k^{-1} = \dfrac{pL - 2\tanh(pL/2)}{p|S|} = \dfrac{y(x=L)}{F}$?

S < 0 (compression)

$\rightarrow$ $k^{-1} = \dfrac{-pL + 2\tan(pL/2)}{p|S|} = \dfrac{y(x=L)}{F}$

force

where $p = \sqrt{\dfrac{|S|}{EI_z}}$



difference in length between inner & outer beams → stress

Determine inner & outer beam stiffnesses:

→ first, determine S

① If polysi strain is ϵ_r, then shoulder expands by $\Delta L_s = \epsilon_r L_s$

② This then applied a load to the beams, $\Delta L = \Delta L_s$

③ Beam Stress:

$$\epsilon_b = \frac{\Delta L}{2L} = \frac{\Delta L_s}{2L} = \pm \epsilon_r \frac{L_s}{2L}$$

Stress Force: $S = \pm E\epsilon_r \left(\frac{L_s}{2L}\right) Wh$ (axial tension)

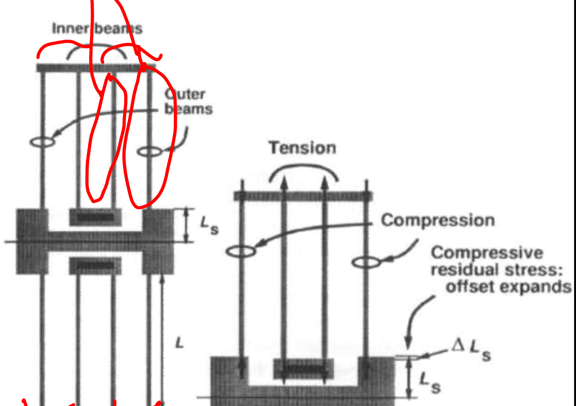
④ Spring Constants:

$$k = 4\left(k_{com}^{-1} + k_{ten}^{-1}\right)^{-1}$$

series combination of oppositely stressed beams

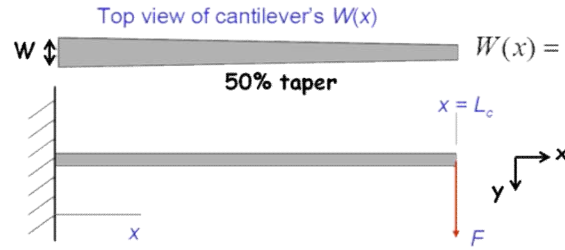
$$k = 4\left[\frac{-pL + 2\tan(pL/2)}{p|S|} + \frac{pL - 2\tanh(pL/2)}{p|S|}\right]^{-1}$$

4 in parallel



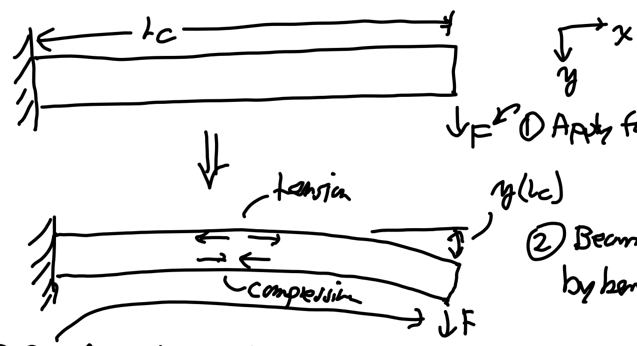
More General Geometries

- Euler-Bernoulli beam theory works well for simple geometries
- But how can we handle more complicated ones?
- Example: tapered cantilever beam
- Objective: Find an expression for displacement as a function of location x under a point load F applied at the tip of the free end of a cantilever with tapered width W(x)



$$W(x) = W\left(1 - \frac{x}{2L_c}\right)$$

Same problem as before: Take a beam, apply force.



① Apply force

② Beam responds by bending.

③ This force does work: $\mathcal{W} = F \cdot y(L_c)$

④ Strain generated (tension & compression)

↳ So the beam has received an inflow of energy, which it stores → stored energy

↳ magnitude of " " determined by shape.

⑤ Then:

$U = \text{Stored Energy} - \underset{\uparrow}{\text{Work Done}} \longrightarrow 0$

when we choose the right shape

This is how we got the beam's response to F.

transfer fcn $y(x) = f(x)$

Fundamentals of Energy Density

General Definition of Work:

$$W(q_1) = \int_0^{q_1} e(q)\,dq$$

q = displacement
e = effort

for EE: $W(Q) = \int_0^Q \frac{Q}{C}\,dQ$

charge

Strain Energy Density

value of strain @ position (x,y,z)

$$w = \int_0^{\epsilon_x} \sigma_x\,d\epsilon_x$$

$\sigma_x(\epsilon_x)$ ← relates stress to strain @ position (x,y,z)

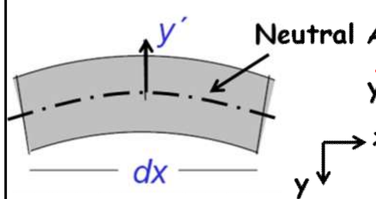
$[\sigma_x = E\epsilon_x]$ →

$$w = \int_0^{\epsilon_x} E\epsilon_x\,d\epsilon_x = \frac{1}{2}E\epsilon_x^2$$

Total Strain Energy [J]:

$$W = \iiint\left\{\frac{1}{2}E\left(\epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2\right) + \frac{1}{2}G\left(\gamma_{xy}^2 + \gamma_{xz}^2 + \gamma_{yz}^2\right)\right\}dV$$

Bending Energy Density



same as z-before

y(x) = transverse displacement of neutral axis

First, find the bending energy dW_{bend} in an infinitesimal length dx:

$$dW_{bend} = \overset{\text{width}}{W}dx \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{2}E\,\underbrace{\epsilon_x^2(y')}_{\text{"fcn of"}}\,dy'$$

$$\left[\frac{1}{R} = \frac{d^2y}{dx^2},\ \epsilon_x = \frac{y'}{R}\right] \rightarrow \epsilon_x(y') = y'\frac{d^2y}{dx^2}$$

$$dW_{bend} = W dx \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{2}E\left[y'\frac{d^2y}{dx^2}\right]^2 dy'$$

$$= \frac{E}{2}\underbrace{\left(\frac{Wh^3}{12}\right)}_{I_z}\left(\frac{d^2y}{dx^2}\right)^2 dx$$

$$\therefore \boxed{W_{bend} = \frac{1}{2}EI_z\int_0^L\left(\frac{d^2y}{dx^2}\right)^2 dx}$$