Lecture 17: Resonance Frequency

- Announcements:
  - Module 9 on Energy Methods online
  - Module 10 on Resonance Frequency online soon
  - Midterm Exam, Thursday, March 17, 5:10-7 p.m. in 3107 Etcheverry
  - Passed out old midterm solutions and went through midterm info sheet last time - the info sheet is in the Quizzes/Midterm/Final link
  - Graded homework in tray outside my office

- Reading: Senturia, Chpt. 10
- Lecture Topics:
  - Energy Methods
  - Virtual Work
  - Energy Formulations
  - Tapered Beam Example
  - Estimating Resonance Frequency

- Reading: Senturia, Chpt. 10: §10.5, Chpt. 19
- Lecture Topics:
  - Estimating Resonance Frequency
  - Lumped Mass-Spring Approximation
  - ADXL-50 Resonance Frequency
  - Distributed Mass & Stiffness
  - Folded-Beam Resonator
  - Resonance Frequency Via Differential Equations

- Last Time:
  - Started energy-based solutions to bending problems

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More General Geometry

- Euler-Bernoulli beam theory works well for simple geometries
- But how can we handle more complicated ones?
- Example: tapered cantilever beam

Objective: Find an expression for displacement as a function of location $x$ under a point load $F$ applied at the tip of the free end of a cantilever with tapered width $W(x)$

$$W(x) = W(1 - \frac{x}{2L_e})$$

Same problem as before: Take a beam, apply force.

1. Strain energy (tension & compression)
   - So the beam has received an input of energy, which it stores as stored energy
   - Magnitude determined by shape.
Then:
\[ U: \text{Stored Energy} - \text{Work Done} \to 0 \]

when we choose the right shape
This is how we get the beam's response to \( F \).

**Fundamentals of Energy Density**

**General Definition of Work:**

\[ W(q) = \int_0^q e(q) \, dq \quad q = \text{displacement} \]

\[ e = \text{elastic} \]

\[ \text{force:} \quad W(q) = \int_0^q \frac{d}{dq} \, dq \]

**Strain Energy Density**: reduced strain @ position \((x, y, z)\)

\[ \omega = \int_0^x \varepsilon \, dx \]

\[ \varepsilon = \frac{\sigma}{E} \]

\[ \left[ \sigma_x, \varepsilon_x \right] \]

\[ \omega = \int_0^x \epsilon_x \, dx = \frac{1}{2} \varepsilon_x^2 \]

**Total Strain Energy** \( (U) \):

\[ U = \int \left\{ \frac{1}{2} E (\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2) \right\} \, dV \]

**Bending Energy Density**

\[ \text{Neutral Axis} \]

\[ y(x) = \text{transverse displacement of neutral axis} \]

First, find the bending energy \( dW_{\text{bend}} \) in an
infinitesimal length \( dx \):

\[ dW_{\text{bend}} = W dx \left[ \frac{1}{2} \frac{1}{2} E \epsilon_x^2 (y') dy' \right] \]

\[ \left[ \frac{1}{R} = \frac{d^2 y}{dx^2}, \epsilon_x = \frac{y''}{R} \right] \rightarrow \epsilon_x(y') = y' \frac{d^2 y}{dx^2} \]

\[ dW_{\text{bend}} = \frac{1}{2} \frac{1}{2} E \left[ y' \frac{d^2 y}{dx^2} \right]^2 dy' \]

\[ = \frac{E}{2} \left( \frac{Wh^3}{12} \right) \frac{d^2 y}{dx^2} dx \]

\[ = \frac{EI_x}{2} \int \frac{d^2 y}{dx^2}^2 \, dx \]
Energy Due to Axial Load

\[ ds = \sqrt{(dx)^2 + (dy)^2} \]

\[ \epsilon_x = \frac{ds - dx}{dx} \approx \frac{1}{2} \left( \frac{dy}{dx} \right)^2 \]

\[ W_{axial} = \frac{1}{2} \int_0^L \left( \frac{dy}{dx} \right)^2 dx \]

\[ \text{Axial Strain Energy} \]

- Look at shear strain energy in your module.

- Go to Module 9, pg. 10, and look at shear strain energy.

- Then, finish off Module 9

Estimating Resonance Frequency

Potential Energy

\[ U(t) = \frac{1}{2} k x(t)^2 = \frac{1}{2} k x_0^2 \cos^2 \omega t \]

Kinetic Energy

\[ K(t) = \frac{1}{2} M \dot{x}^2(t) = \frac{1}{2} M x_0^2 \omega^2 \sin^2 \omega t \]

Remarks:

1. Energy must be conserved.
The proof mass of the ADXL-50 is many times larger than the effective mass of its suspension beams. Can ignore the mass of the suspension beams (which greatly simplifies the analysis).

- Suspension Beam: $L = 260 \, \mu m$, $h = 2.3 \, \mu m$, $W = 2 \, \mu m$

In fabrication, purposely introduce a tensile stress on the beams:

a fairly large one $\rightarrow$ okay for an air bag accelerometer.

$\omega_0 = \sqrt{\frac{k}{m}}$ 

$\Rightarrow$ good to problem when mass stiffness can be segregated; i.e., when they are distinct.
Bending Contribution

\[ k_b = \frac{1}{L} \left( \frac{1}{L} \right)^3 = \frac{L^3}{E h w^3} \]

Stretches Contribution

\[ k_s = \frac{1}{2} \frac{L}{h} \]

\[ F_y = \sigma s \theta = \sigma s \left( \frac{L}{L} \right) = \sigma s \]

\[ \text{Maximum displacement function (i.e., mode shape function)} \]
\[ \hat{y}(x) \]

To get the total spring constant

\[ k = 4 (k_b + k_s) = 4 (0.24 + 0.88) = 4.5 \, \text{N/m} \]

Now, get resonance freq:

\[ f_0 = \frac{1}{2 \pi} \sqrt{\frac{k}{m}} = \frac{1}{2 \pi} \sqrt{\frac{4.5 \, \text{N/m}}{1.2 \times 10^{-4} \, \text{kg}}} = 26.5 \, \text{kHz} \]

ADXL50 Data Sheet: \( f_0 = 24 \, \text{kHz} \)

\[ \text{Capacitive transducer} \]
\[ \text{Electrical stiffness} \]

\[ \text{Find the resonance frequency when mass + stiffness are distributed} \]

\[ y(x, t) = \hat{y}(x) \cos(\omega t) \]

\[ \text{Maximum displacement function} \]
\[ \hat{y}(x) \]

\[ \text{Velocity topographical mapping} \]

\[ y(x, t) = 0 \]