Lecture 17: Resonance Frequency

- **Announcements**:
- Module 9 on Energy Methods online
- Module 10 on Resonance Frequency online soon
- Midterm Exam, Thursday, March 17, 5:10-7 p.m. in 3107 Etcheverry
- Passed out old midterm solutions and went through midterm info sheet last time – the info sheet is in the Quizzes/Midterm/Final link
- Graded homework in tray outside my office
- ------------------------------------
- Reading: Senturia, Chpt. 10
- Lecture Topics:
  - Energy Methods
  - Virtual Work
  - Energy Formulations
  - Tapered Beam Example
  - Estimating Resonance Frequency
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- Reading: Senturia, Chpt. 10: §10.5, Chpt. 19
- Lecture Topics:
  - Estimating Resonance Frequency
  - Lumped Mass-Spring Approximation
  - ADXL-50 Resonance Frequency
  - Distributed Mass & Stiffness
  - Folded-Beam Resonator
  - Resonance Frequency Via Differential Equations
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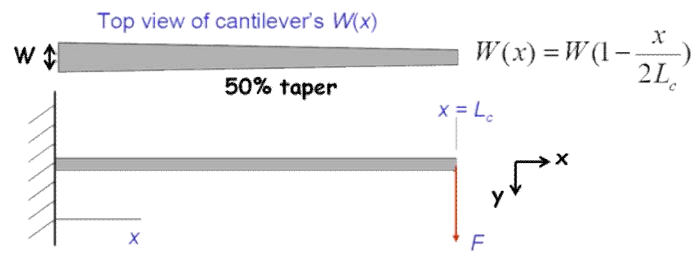
- **Last Time**:
- Started energy-based solutions to bending problems

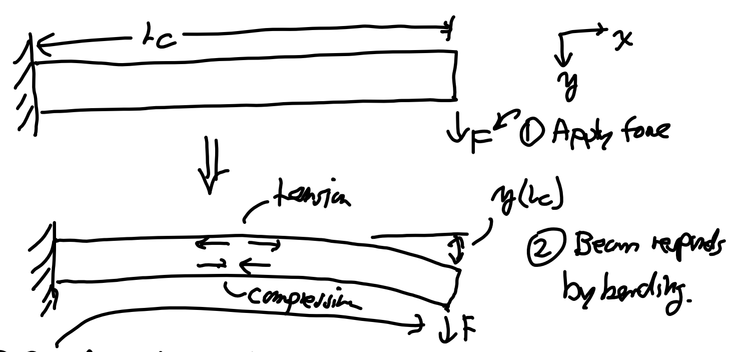
More General Geometries

- Euler-Bernoulli beam theory works well for simple geometries
- But how can we handle more complicated ones?
- Example: tapered cantilever beam
- Objective: Find an expression for displacement as a function of location x under a point load F applied at the tip of the free end of a cantilever with tapered width W(x)



$$W(x) = W\left(1 - \frac{x}{2L_c}\right)$$

Same problem as before: Take a beam, apply force.



③ This force does work: $W = F\cdot y(L_c)$

④ Strain generated (tension & compression)

↳ So the beam has received an influx of energy, which it stores → stored energy

↳ magnitude of " " determined by shape.

⑤ Then:

U: Stored Energy − Work Done $\longrightarrow 0$

when we choose the right shape

This is how we got the beam's response to F.

transfer fcn $y(x) = f(x)$

Fundamentals of Energy Density

General Definition of Work:

$$W(q_1) = \int_0^{q_1} e(q)\,dq$$

q = displacement
e = effort

for EE: $W(Q) = \int_0^{Q} \frac{Q}{C}\,dQ$

charge

Strain Energy Density]

value of strain @ position (x,y,z)

$$w = \int_0^{\epsilon_x} \sigma_x \, d\epsilon_x$$

$\sigma_x(\epsilon_x)$ ← relates stress to strain @ position (x,y,z)

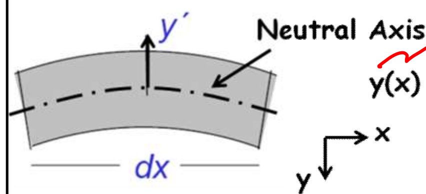
$[\sigma_x = E\epsilon_x]$

$$w = \int_0^{\epsilon_x} E\epsilon_x \, d\epsilon_x = \frac{1}{2}E\epsilon_x^2$$

Total Strain Energy [J]:

$$W = \iiint \left\{ \frac{1}{2}E(\epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2) + \frac{1}{2}G(\gamma_{xy}^2 + \gamma_{xz}^2 + \gamma_{yz}^2) \right\} dV$$

Bending Energy Density



same as z-before

y(x) = transverse displacement of neutral axis

First, find the bending energy dW_{bend} in an infinitesimal length dx:

width

$$dW_{bend} = W dx \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{2} E \epsilon_x^2(y')\, dy'$$

"fcn of"

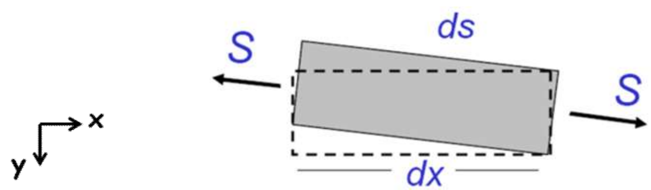
$$\left[\frac{1}{R} = \frac{d^2y}{dx^2},\ \epsilon_x = \frac{y'}{R}\right] \rightarrow \epsilon_x(y') = y'\frac{d^2y}{dx^2}$$

$$dW_{bend} = W dx \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{2} E \left[y' \frac{d^2y}{dx^2}\right]^2 dy'$$

$$= \frac{E}{2}\underbrace{\left(\frac{Wh^3}{12}\right)}_{I_z}\left(\frac{d^2y}{dx^2}\right)^2 dx$$

$$\therefore \boxed{W_{bend} = \frac{1}{2}EI_z \int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx}$$

Energy Due to Axial Load



⇒ energy related to lengthening

$$ds = [(dx)^2 + (dy)^2]^{1/2} = dx\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}$$

binomial theorem $\simeq dx\left[1+\frac{1}{2}\left(\frac{dy}{dx}\right)^2\right]$

$$\therefore \epsilon_x = \frac{ds-dx}{dx} = \frac{1}{2}\left(\frac{dy}{dx}\right)^2$$

$$\left[dW_{axial} = S\epsilon_x dx = \frac{1}{2} S \left(\frac{dy}{dx}\right)^2 dx\right]$$

$$\boxed{W_{axial} = \frac{1}{2} S \int_0^L \left(\frac{dy}{dx}\right)^2 dx}$$

Axial Strain Energy

⇒ Look @ shear strain energy in your module.

- Go to Module 9, pg. 10, and look at shear strain energy
- Then, finish off Module 9

Estimating Resonance Frequency

$x(t) = X_0 \cos\omega t$

Potential Energy

$$W(t) = \frac{1}{2} k x^2(t) = \frac{1}{2} k X_0^2 \cos^2\omega t$$

Kinetic Energy

$$K(t) = \frac{1}{2} M \dot{x}^2(t) = \frac{1}{2} M X_0^2 \omega^2 \sin^2\omega t$$

$\dot{x} = \frac{dx}{dt}$ = velocity

Remarks.

① Energy must be conserved.

② Total Energy = Potential Energy + Kinetic Energy at all times & locations on the structure

$$W_{max} = \frac{1}{2} k X_0^2 = K_{max} = \frac{1}{2} M\omega^2 X_0^2$$

maximum potential energy

peak displacement

radian freq.

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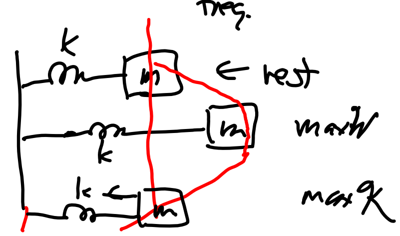
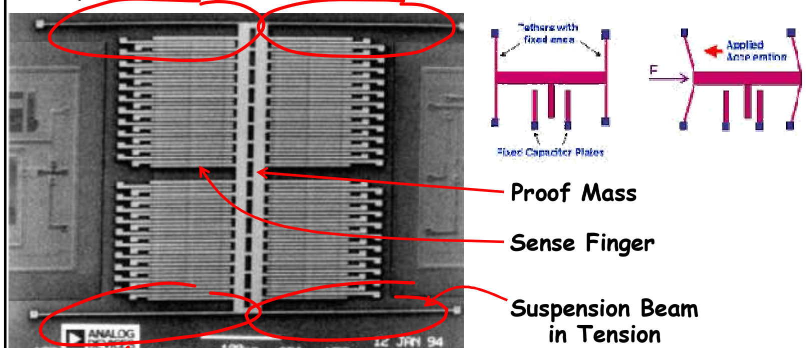


$$\boxed{\omega_0 = \sqrt{\frac{K}{M}}}$$

⇒ good for problems where mass & stiffness can be separated; i.e., where they are distinct.

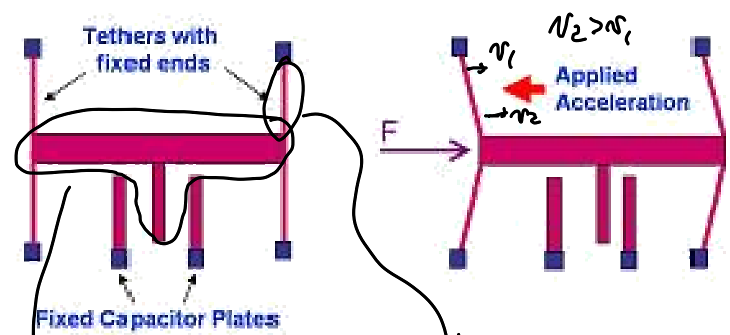
Good Example: ADXL50 Accelerometer

- The proof mass of the ADXL-50 is many times larger than the effective mass of its suspension beams
  - Can ignore the mass of the suspension beams (which greatly simplifies the analysis)
- Suspension Beam: L = 260 μm, h = 2.3 μm, W = 2 μm



In fabrication: purposely introduce a tensile stress in the beams!

a fairly large one → okay, for an air bag accelerometer



Mass of ~~structure~~ proof mass >> mass of springs

∴ ignore the mass of the springs

Stiffness of ~~structure~~ proof mass >> stiff. of springs

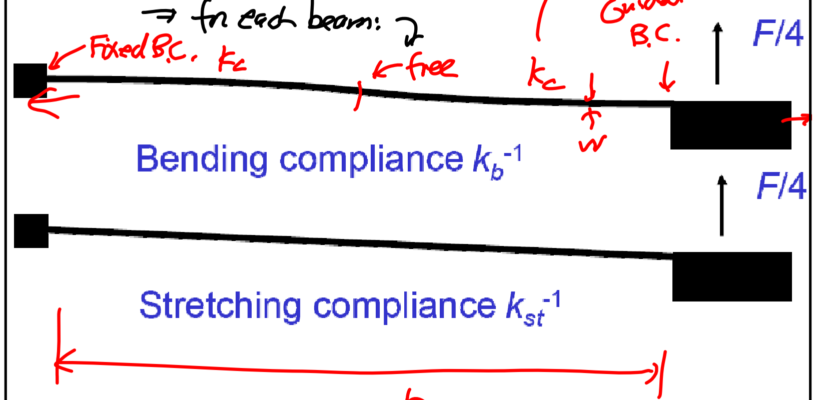
∴ ignore the stiffness of the ~~structure~~ proof mass

For the ADXL-50: M = 162 ng

(60% of the mass from fingers)

Suspension: Four tensioned beams

→ for each beam: ?



Bending Contribution

$$k_b^{-1} = \left(\frac{1}{k_c}+\frac{1}{k_c}\right) = 2\left(\frac{(L/2)^3}{3E(hW^3/12)}\right) = \frac{L^3}{EhW^3}$$

length, width, thickness

$= 4.2$ m/N

Stretching Contribution

$$F_y = S\sin\theta \approx S\theta \approx S\left(\frac{y}{L}\right) = \left(\frac{S}{L}\right)y$$

[assume small displacements]

k_{st} — stretching stiffness

$$k_{st}^{-1} = \frac{L}{S} = \frac{L}{\sigma_r W h} = 1.14 \text{ m/N}$$

To get the total spring constant
↳ add the bending & stretching stiffnesses

$$k = 4(k_b + k_{st}) = 4(0.24+0.88) = \underline{4.5 \text{ N/m}}$$

Now, get resonance freq.:

$$f_o = \frac{1}{2\pi}\sqrt{\frac{K}{M}} = \frac{1}{2\pi}\sqrt{\frac{4.48 \text{ N/m}}{162\times10^{-9}\text{ kg}}} = 26.5\text{ kHz}$$

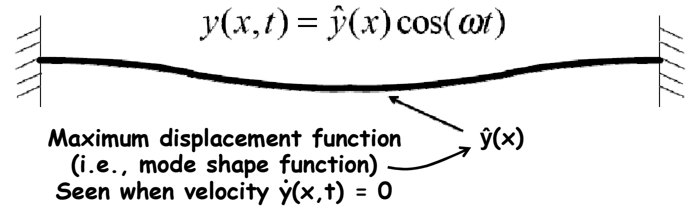
ADXL-50 Data Sheet: $f_o = 24$ kHz — difference?

↳ Capacitive transducer

↳ electrical stiffness!

Find the Resonance Frequency When Mass & Stiffness are Distributed

- Vibrating structure displacement function:



- Procedure for determining resonance frequency:
  - Use the static displacement of the structure as a trial function and find the strain energy W_{max} at the point of maximum displacement (e.g., when t=0, π/ω, …)
  - Determine the maximum kinetic energy when the beam is at zero displacement (e.g., when it experiences its maximum velocity)
  - Equate energies and solve for frequency