Lecture 19: Equivalent Circuits I

- Announcements:
- Module 10 on Resonance Frequency online
- Module 11 on Equivalent Circuits online
- HW#5 online and due Wednesday, April 13
- Project described last time (info online)
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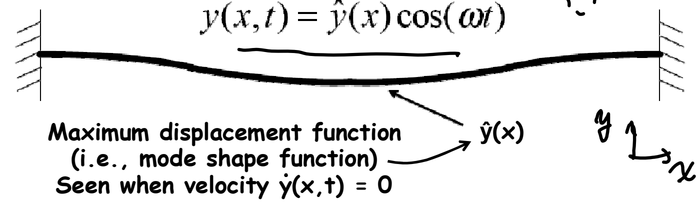
- Reading: Senturia, Chpt. 10: §10.5, Chpt. 19
- Lecture Topics:
  - Estimating Resonance Frequency
  - Lumped Mass-Spring Approximation
  - ADXL-50 Resonance Frequency
  - Distributed Mass & Stiffness
  - Folded-Beam Resonator
  - Resonance Frequency Via Differential Equations
- --
- Reading: Senturia, Chpt. 5
- Lecture Topics:
  - Lumped Mechanical Equivalent Circuits
  - Electromechanical Analogies
- --
- Last Time:

over

Find the Resonance Frequency When Mass & Stiffness are Distributed

- Vibrating structure displacement function:

$$y(x,t) = \hat{y}(x)\cos(\omega t)$$

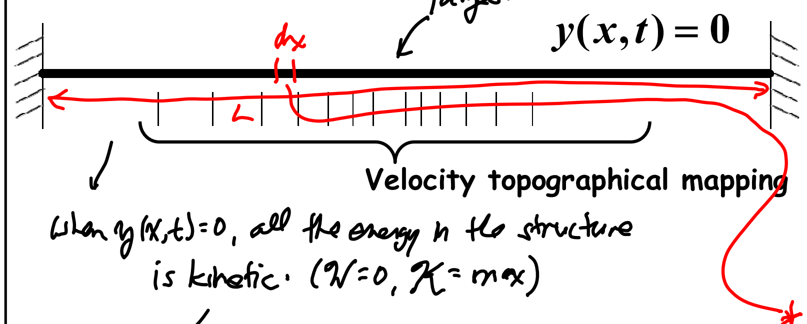


- Procedure for determining resonance frequency:
  - Use the static displacement of the structure as a trial function and find the strain energy W_{max} at the point of maximum displacement (e.g., when t=0, π/ω, …)
  - Determine the maximum kinetic energy when the beam is at zero displacement (e.g., when it experiences its maximum velocity)
  - Equate energies and solve for frequency

Get Maximum Kinetic Energy!

velocity: $v(x,t) = \frac{\partial y(x,t)}{\partial t} = -\omega \hat{y}(x)\sin(\omega t)$

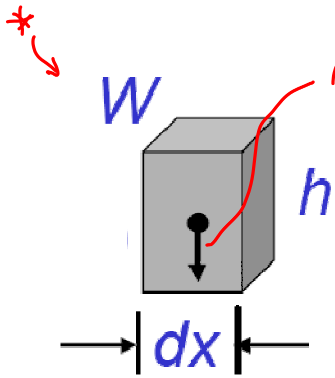
largest velocity



When $y(x,t)=0$, all the energy in the structure is kinetic. ($W=0$, $K=max$)

$$v\left(x, \frac{(2m+1)\pi}{2\omega}\right) = -\omega \hat{y}(x)$$

$t = \frac{\pi}{2\omega}, \frac{3\pi}{2\omega}, \dots$



velocity: $v = -\omega \hat{y}(x)$ ← mode shape, $\frac{(2m-1)\pi}{2W}$

$dK = \frac{1}{2} \cdot dm \cdot [v(x,t)]^2$

$dm = \rho(Whdx)$ ← density

Maximum K:

$$K_{max} = \int_0^L \frac{1}{2}\rho W h \, dx \, v^2(x,t) = \int_0^L \frac{1}{2}\rho W h \omega^2 [\hat{y}(x)]^2 dx$$

To get frequency:

$K_{max} = W_{max}$

$$\therefore \quad \omega = \sqrt{\frac{W_{max}}{\int_0^L \frac{1}{2}\rho W h [\hat{y}(x)]^2 dx}}$$

ω = radian resonance frequency

W_{max} = maximum potential energy

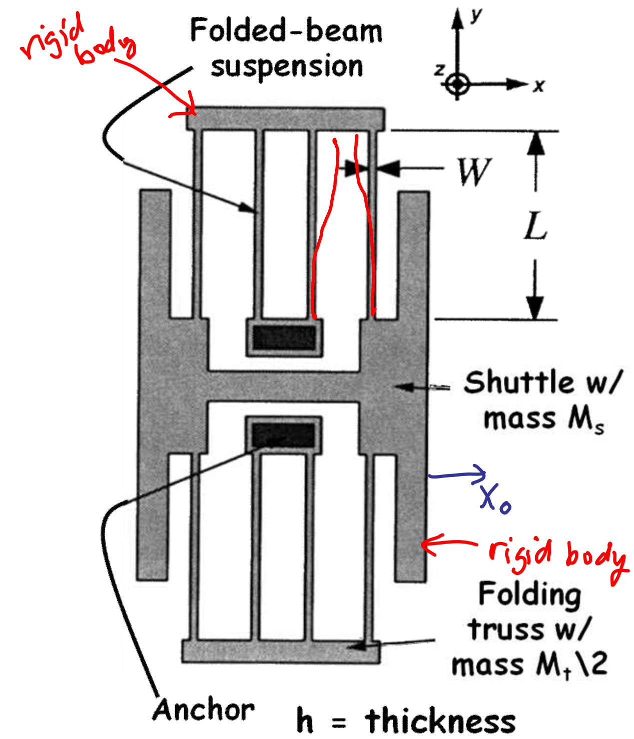
ρ = density of the structural material

W = beam width

h = " thickness

$\hat{y}(x)$ = resonance mode shape

Resonance Freq. of a Folded-Beam Resonator



- Derive an expression for the resonance frequency of the above structure

Use the Rayleigh-Ritz Method: (energy method)

$K_{max} = W_{max}$

Find the kinetic energy → one piece @ a time:

$$K_{max} = \underbrace{K_s}_{\text{shuttle}} + \underbrace{K_t}_{\text{truss}} + \underbrace{K_b}_{\text{beams}}$$

$$\mathcal{K}_{max} = \frac{1}{2} v_s^2 M_s + \frac{1}{2} v_t^2 M_t + \frac{1}{2}\int v_b^2 \, dM_b$$

Velocity of Shuttle: $v_s = \omega_o X_o$

resonance freq.

max. displacement of the shuttle

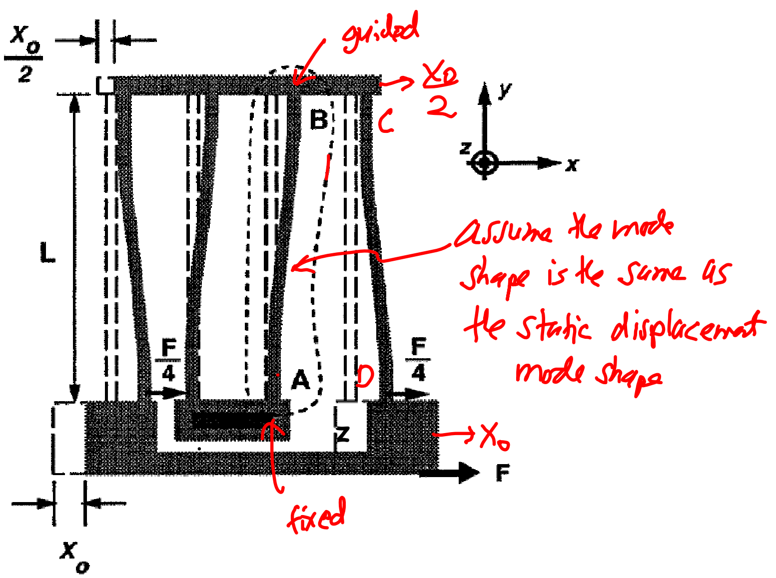
$$\therefore \mathcal{K}_s = \frac{1}{2} v_s^2 M_s = \boxed{\frac{1}{2}\omega_o^2 X_o^2 M_s}$$

Velocity of Truss: $v_t = \frac{1}{2} v_s = \frac{1}{2}\omega_o X_o$

$$\therefore \mathcal{K}_t = \frac{1}{2}\left(\frac{1}{2}\omega_o X_o\right)^2 M_t = \frac{1}{8}\omega_o^2 X_o^2 M_t$$

mass of both trusses combined

Velocity of the Beam Segments:



Segment [AB]:

$$\hat{x}(y) = \frac{F_x}{48EI_z}\left(3Ly^2 - 2y^3\right), \quad 0 \le y \le L \qquad (1)$$

At $y = L$: $x(L) = \frac{X_o}{2} = \frac{F_x L^3}{48EI_z}$ ← B.C.

Substitute into (1):

$$\hat{x}(y) = \frac{X_o}{2}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]$$

Which yields for velocity:

$$v_b(y)\big|_{[AB]} = \frac{X_o}{2}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]\omega_o$$

Plugging into the expression for $\mathcal{K}_b$:

$$\mathcal{K}_b\big|_{[AB]} = \frac{1}{2}\int_0^L \frac{X_o^2\omega_o^2}{4}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]^2 dM_{[AB]}$$

$$= \frac{X_o^2\omega_o^2 M_{[AB]}}{8L}\int_0^L \left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]^2 dy$$

$M_{[AB]}$ = static mass

mass per unit length

$$\boxed{\mathcal{K}_b\big|_{[AB]} = \frac{13}{280} X_o^2 \omega_o^2 M_{[AB]}}$$

For segment [CD]:

$$v_b(y)\big|_{[CD]} = X_o\left[1 - \frac{3}{2}\left(\frac{y}{L}\right)^2 + \left(\frac{y}{L}\right)^3\right]\omega_o$$

Thus:

$$\mathcal{K}_b\big|_{[CD]} = \frac{x_o^2\omega_o^2 M_{[CD]}}{2L}\int_0^L \left[1-\frac{3}{2}\left(\frac{y}{L}\right)^2+\left(\frac{y}{L}\right)^3\right]^2 dy$$

$$\mathcal{K}_b\big|_{[CD]} = \frac{83}{280}x_o^2\omega_o^2 M_{[CD]}$$

static mass of beam [CD]

Let $M_b \triangleq$ total mass of all 8 beams.

Then $M_{[AB]} = M_{[CD]} = \frac{1}{8}M_b$

Thus:

$$\mathcal{K}_b = 4\mathcal{K}_b\big|_{[AB]} + 4\mathcal{K}_b\big|_{[CD]} = \frac{6}{35}x_o^2\omega_o^2 M_b$$

and

$$\mathcal{K}_{max} = x_o^2\omega_o^2\left[\frac{1}{2}M_s + \frac{1}{8}M_t + \frac{6}{35}M_b\right]$$

Get $\mathcal{W}_{max}$ → simply equal to the work done to achieve maximum deflection

$$\mathcal{W}_{max} = \frac{1}{2}k_x x_o^2$$

for a folded-beam device as shown is just $k_{cantilever}$ $= k_c$

Thus, using Rayleigh-Ritz:

$$\mathcal{K}_{max} = \mathcal{W}_{max}$$

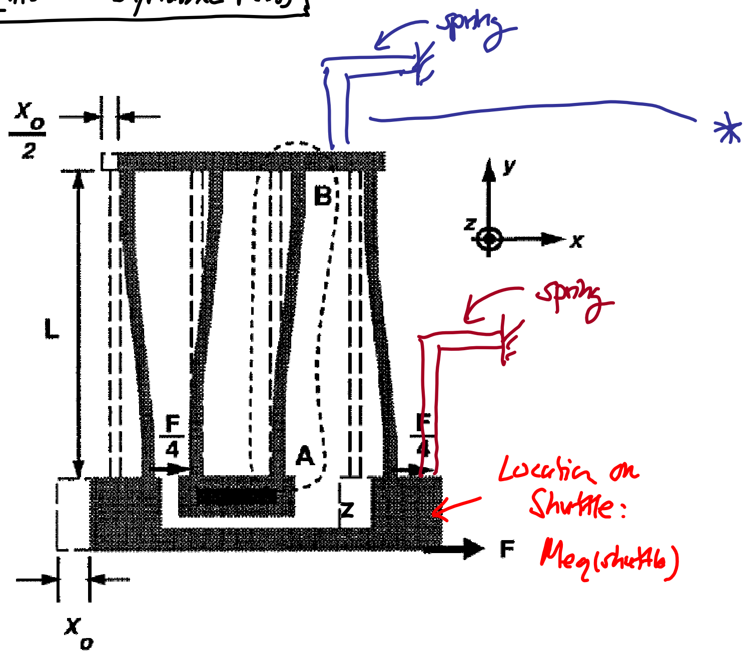
$$\cancel{x_o^2}\omega_o^2\left[\frac{1}{2}M_s + \frac{1}{8}M_t + \frac{6}{35}M_b\right] = \frac{1}{2}k_c \cancel{x_o^2}$$

$$\boxed{\omega_o = \left[\frac{k_c}{M_{eq}}\right]^{1/2} \quad \text{where } M_{eq} = M_s + \frac{1}{4}M_t + \frac{12}{35}M_b}$$

(Resonance Freq. of a Folded-Beam Suspended Shuttle)

- Go through slides 21-31 of Module 10

Equivalent Dynamic Mass



Equivalent Mass:

$$\text{Equiv. Mass} = M_{eq,x} = \frac{K_{max}}{\frac{1}{2}V_x^2} = \frac{\frac{1}{2}\rho A \int_0^{\ell} V^2(x)\,dx}{\frac{1}{2}V_x^2}$$

V_x ← velocity @ location x

$$M_{eq(shuttle)} = \frac{K_{max}}{\frac{1}{2}V_{shuttle}^2} = \frac{\cancel{\omega_o^2 x_o^2 \left(\frac{1}{2}\right)}\left[M_s + \frac{1}{4}M_t + \frac{12}{35}M_b\right]}{\cancel{\frac{1}{2}\omega_o^2 x_o^2}}$$

$$M_{eq(shuttle)} = M_s + \frac{1}{4}M_t + \frac{12}{35}M_b$$

*

$$M_{eq(truss)} = \frac{K_{max}}{\frac{1}{2}V_{truss}^2} = \frac{\omega_o^2 x_o^2 \left(\frac{1}{2}\right)\left[M_s + \frac{1}{4}M_t + \frac{12}{35}M_b\right]}{\cancel{\frac{1}{2}}\left(\frac{1}{4}\right)\omega_o^2 x_o^2}$$

$$M_{eq(truss)} = 4\left[M_s + \frac{1}{4}M_t + \frac{12}{35}M_b\right]$$

Equiv. Dynamic Mass

Equiv. Dynamic Stiffness

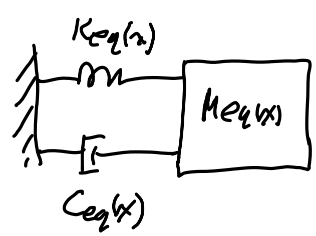
$$\omega_o = \sqrt{\frac{K_{eq}(x)}{M_{eq}(x)}} \longrightarrow \boxed{K_{eq}(x) = \omega_o^2 M_{eq}(x)}$$

⇒ large equiv. mass → large equiv. stiffness
(they go hand in hand)

Equiv. Dynamic Damping

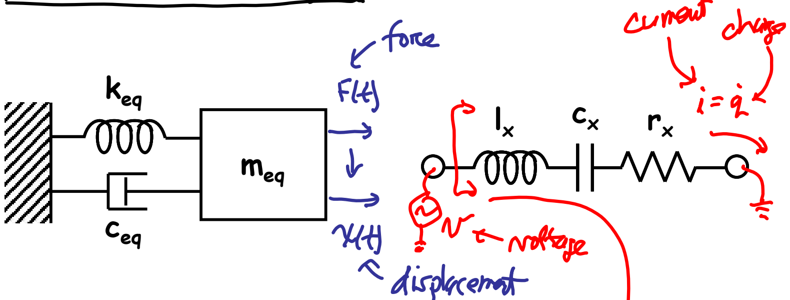
$$Q = \frac{\omega_o M_{eq}(x)}{C_{eq}(x)} \sim \frac{L}{R} \longrightarrow \boxed{C_{eq}(x) = \frac{\omega_o M_{eq}(x)}{Q} = \frac{\sqrt{K_{eq}(x) M_{eq}(x)}}{Q}}$$

↑ damping



} specified @ a single location x

Electromechanical Analogies



$F(t) = F\cos(\omega t) \rightarrow x(t) = X\cos(\omega t)$
(off resonance)

Equation of Motion:

$$m_{eq}\ddot{x} + c_{eq}\dot{x} + k_{eq}x = F(t)$$

⇒ using phasor concepts:

$$F = j\omega m_{eq}\dot{x} + \frac{k_{eq}}{j\omega}\dot{x} + c_{eq}\dot{x}$$

Impedance looking in:

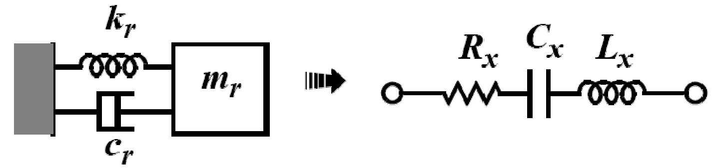
$$\frac{v}{i} = j\omega l_x + \frac{1}{j\omega c_x} + r_x$$

$$v = j\omega l_x i + \frac{(1/c_x)}{j\omega} i + r_x i$$

⇒ by analogy:

$$\boxed{F \rightarrow v \quad m_{eq} \rightarrow l_x \quad c_{eq} \rightarrow r_x \quad \dot{x} \rightarrow i \quad k_{eq} \rightarrow \frac{1}{c_x}}$$

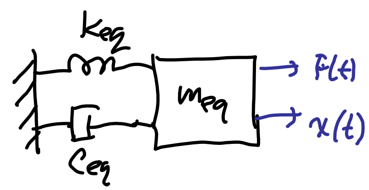
[Parameter Relationships in the Current Analogy]



- Mechanical-to-electrical correspondence in the current analogy:

Mechanical Variable	Electrical Variable
Damping, c	Resistance, R
Stiffness^{-1}, k^{-1}	Capacitance, C
Mass, m	Inductance, L
Force, f	Voltage, V
Velocity, v	Current, I

Lowpass Biquad Transfer Function



$$F = j\omega m_{eq}\dot{x} + \frac{k_{eq}}{j\omega}\dot{x} + c_{eq}\dot{x}$$

⇒ convert to full phasor form:

$$F = (j\omega X)(j\omega X)m_{eq} + \frac{k_{eq}}{\cancel{j\omega}}(\cancel{j\omega}X) + c_{eq}(j\omega X)$$

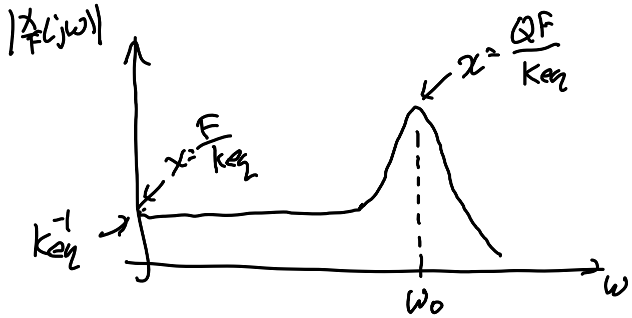
$$\frac{X}{F}(j\omega) = \frac{1}{k_{eq}}\left[-\omega^2\frac{m_{eq}}{k_{eq}} + 1 + j\frac{c_{eq}\omega}{k_{eq}}\right]^{-1}$$

$$\left[\frac{k_{eq}}{m_{eq}} = \omega_o^2,\ Q = \frac{m_{eq}\omega_o}{c_{eq}} = \frac{k_{eq}}{\omega_o c_{eq}} \rightarrow \frac{k_{eq}}{c_{eq}} = Q\omega_o\right]$$

*

$$\frac{X}{F}(j\omega) = \frac{1}{k_{eq}}\left[-\left(\frac{\omega}{\omega_0}\right)^2 + 1 + j\frac{\omega}{Q\omega_0}\right]^{-1}$$

$$\frac{X}{F}(j\omega) = \frac{k_{eq}^{-1}}{1-\left(\frac{\omega}{\omega_0}\right)^2 + j\frac{\omega}{Q\omega_0}}$$



- Go through pages 11-22 of Module 11