Lecture 20: Capacitive Transducers

- Announcements:
- Module 12 on Capacitive Transducers online
- HW#5 online and due Wednesday, April 13
- Project slide #1 due Friday, April 8
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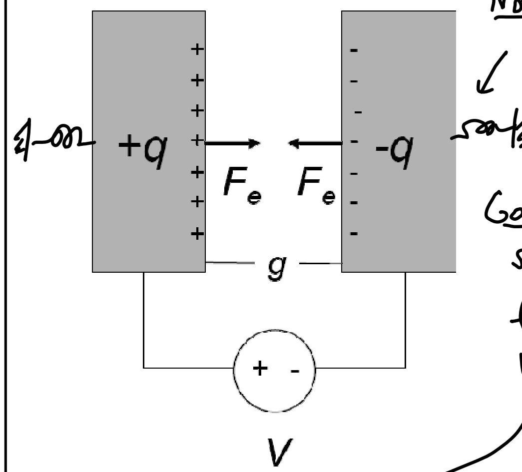
- Reading: Senturia, Chpt. 5, Chpt. 6
- Lecture Topics:
  - Energy Conserving Transducers
    - Charge Control
    - Voltage Control
  - Parallel-Plate Capacitive Transducers
    - Linearizing Capacitive Actuators
    - Electrical Stiffness
  - Electrostatic Comb-Drive
    - 1st Order Analysis
    - 2nd Order Analysis
- -------------------------------------
- Last Time:
- Started equivalent circuits based on electromechanical analogies
- Specified circuit model for mechanical behavior
- Must still develop a circuit model for the electrical-to-mechanical transducer

over

$\boxed{\text{Basic Physics of Electrostatic Actuation}}$



Note: Assume the plates are supported elastically → can move

Goal: Determine gap spacing g as a function of input variables

1st: Determine the energy of the system.

2nd: Ask: What can I do to Δ the energy of the system?

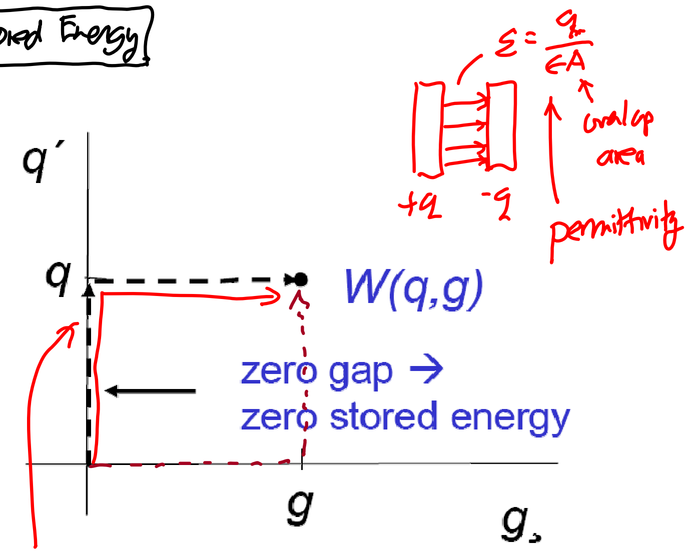
① change the charge q

② change the separation g

$$\Delta W(q,g) = V\Delta q + F_e \Delta g$$

$$dW = Vdq + F_e dg$$

Stored Energy



No change in charge: $dq = 0$

$$W = 0 + \int_0^g F_e \, dg'$$

$$F_e = \left(\frac{q}{2}\right)\mathcal{E} = \frac{1}{2}\frac{q^2}{\epsilon A} \quad \text{(independent of } g\text{)}$$

$$\therefore W = \int_0^g F_e \, dg' = F_e g' \Big|_0^g = F_e g$$

$$\boxed{W(g) = \frac{1}{2}\frac{q^2}{\epsilon A} g} \leftarrow = \;{**}$$

over $\hookleftarrow$ ——— $* \rightarrow$

$* \rightarrow$ Work done to charge C to q at a fixed gap g:

$$dW = V\,dq + F_e \cancel{dg}^{\,0}$$

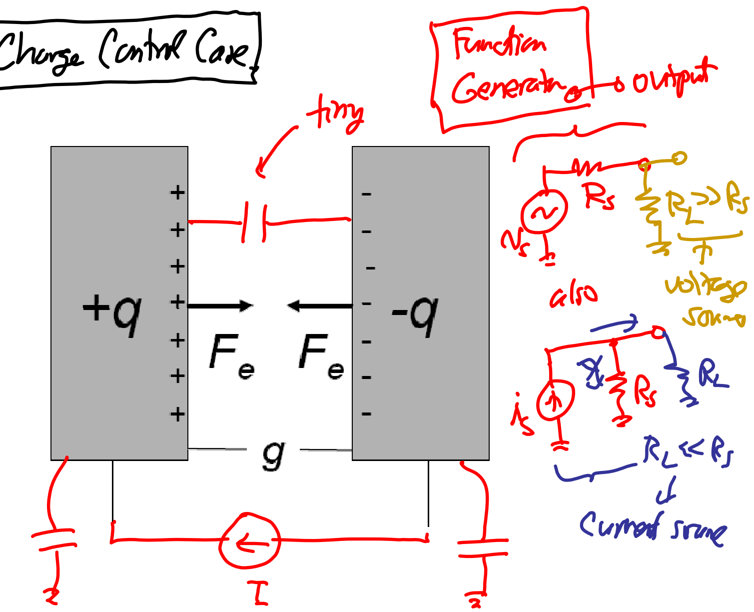
For a capacitor:

$$q = CV \rightarrow V = \frac{q}{C}$$

$$\therefore W(q) = \int_0^q V\,dq' = \int_0^q \left(\frac{q'}{C}\right) dq' = \frac{1}{2}\frac{q^2}{C}$$

$$** \quad \boxed{= \frac{1}{2}\frac{q^2}{\epsilon A} g = W(q)}$$

Charge Control Case



From $dW = V\cancel{dq} + F_e dg$

⇒ Force is given by

$$F_e = \frac{\partial W(q,g)}{\partial g}\bigg|_{q=\text{const.}} = \frac{\partial}{\partial g}\left(\frac{1}{2}\frac{q^2}{\epsilon A}g\right)$$

$$\therefore \boxed{F_e = \frac{1}{2}\frac{q^2}{\epsilon A}}$$ → indep. of gap spacing!

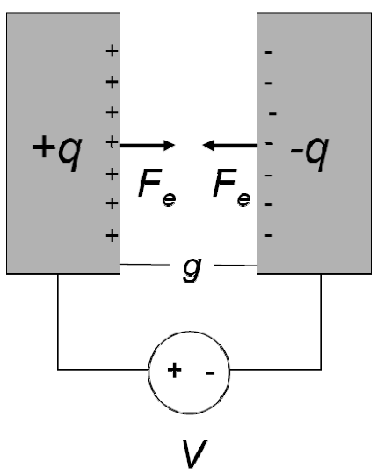
⇒ voltage is given by:

$$V = \frac{\partial W(q,g)}{\partial q}\bigg|_{g=\text{const.}} = \frac{\partial}{\partial q}\left(\frac{1}{2}\frac{q^2}{\epsilon A}g\right)$$

$$= \frac{qg}{\epsilon A} \rightarrow \boxed{V = \frac{q}{C}} \checkmark$$

(consistent w/ what we know)

Voltage Control



Want to write:
$F_e = f(V)$

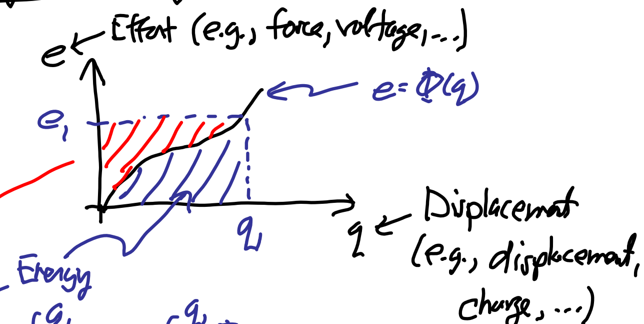
We know this:
$dW = V dq + F_e dg$

$W = W(q,g)$

Need: $W'(V,g)$
↓ ← replace charge q w/ voltage V

Can get this using a Legendre transformation.

Energy & Co-Energy



$$W(q_1) = \int_0^{q_1} e\,dq = \int_0^{q_1} \Phi(q)\,dq$$

Co-Energy:

$$W'(e_1) = \int_0^{e_1} q\,de = \int_0^{e_1} \Phi^{-1}(e)\,de$$

For a linear system, these will be equal.

Can define co-energy as:

$$W'(e) = eq - W(q) \quad \text{(from the plot)}$$

↑ co-energy ↑ energy

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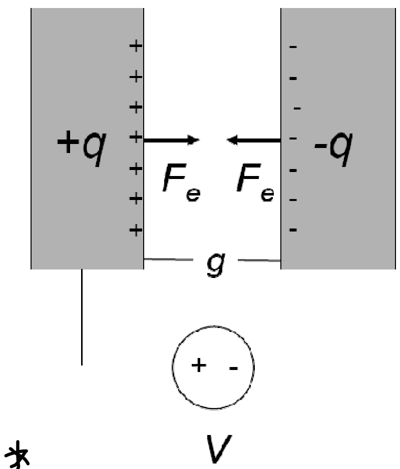
Co-Energy Formulation for Voltage Control



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$$\mathcal{W}'(V,g) = Vq - \mathcal{W}(q,g)$$

Differentially, this becomes

$$d\mathcal{W}'(V,g) = (q\,dV + V\cancel{dq}) - d\mathcal{W}(q,g)$$

$$\left[d\mathcal{W}(q,g) = F_e dg + V\cancel{dq}\right]$$

$$\boxed{d\mathcal{W}'(V,g) = q\,dV - F_e dg} \leftarrow$$ working co-energy expression

Find co-energy in terms of voltage, V:

$$\mathcal{W}' = \int_0^V q(g,V')\,dV' = \int_0^V \left(\frac{\epsilon A}{g}\right) V'\,dV'$$

$$= \frac{1}{2}\left(\frac{\epsilon A}{g}\right)V^2 = \frac{1}{2}CV^2 \checkmark \quad \text{(as expected)}$$

Voltage-Controlled Electrostatic Force:

$$F_e = -\left.\frac{\partial \mathcal{W}'(V,g)}{\partial g}\right)_{V=\text{const.}}$$

$$= -\frac{1}{2}\left(\frac{\epsilon A}{g^2}\right)V^2 = \boxed{\frac{1}{2}\frac{C}{g}V^2 = F_e}$$

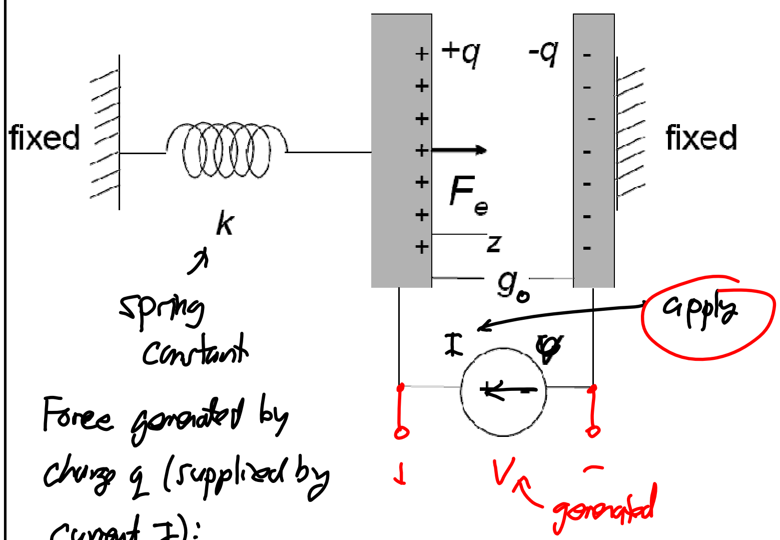
depends on gap!

Charge:

$$q = \left.\frac{\partial \mathcal{W}'(V,g)}{\partial V}\right)_{g=\text{const.}} = \frac{\epsilon A}{g}V = CV \checkmark$$

(as expected)

Charge-Control of a Spring-Suspended C



Force generated by charge q (supplied by current I):

$$F_e = \frac{\partial W(q,g)}{\partial g}\bigg|_q = \frac{q^2}{2\epsilon A}$$

Restoring force of spring: $F_{spring} = kz = F_e$ ← equilibrium

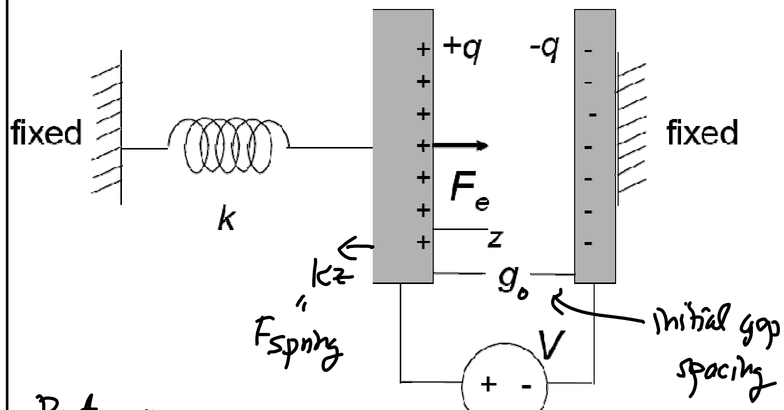
The gap:

$$g = g_o - z = g_o - \frac{F_e}{k} = \boxed{g_o - \frac{1}{2}\frac{q^2}{\epsilon A}\frac{1}{k} = g}$$

↳ q↑ can drive $g \to 0$ in continuous fashion

$$V = \frac{q}{C} = \frac{q}{\epsilon A} g = \boxed{\frac{q}{\epsilon A}\left(g_o - \frac{1}{2}\frac{q^2}{\epsilon A}\frac{1}{k}\right) = V}$$ ← V↓ as g↓

Voltage-Control of a Suspended C



But now:

$$F_e = \frac{\partial W'(V,g)}{\partial g}\bigg|_q \rightarrow F_e = \frac{1}{2}\frac{\epsilon A}{g^2}V^2$$

And the gap:

$$g = g_o - z = g_o - \frac{F_e}{k} = \boxed{g_o - \frac{1}{2}\frac{\epsilon A}{g^2}\frac{V^2}{k} = g}$$

g shows up on both sides!

If V↑ → g↓ → F_e↑ (+) Feedback!

If loop gain > 1, then this will go unstable!

plate will collapse! (into the electrode)

Charge: (for a stable gap)

$$q = \frac{\partial \mathcal{W}'(V,g)}{\partial V}\Big|_g = CV \checkmark \quad \text{(as expected)}$$

Stability Analysis

⇒ determine under what conditions voltage control will cause collapse of the plates:

$$F_{net} = F_e - F_{spring} = \underbrace{\frac{\epsilon A V^2}{2g^2}}_{F_e} - \underbrace{k(g_o - g)}_{spring}$$

What happens when I change g by a small increment dg?

↳ get an increment in the net attractive force F_{net}

$$\underbrace{dF_{net}}_{(-)} = \frac{\partial F_{net}}{\partial g} dg = \left[-\frac{\epsilon A V^2}{g^3} + k\right] \underbrace{dg}_{(-)}$$

If $g\downarrow \rightarrow dg = (-)$, then for stability need $F_{net}\downarrow \rightarrow dF_{net} = (-)$

This must be (+)! → otherwise, the plates collapse!

Thus: $\boxed{k > \frac{\epsilon A V^2}{g^3}}$ (for a stable uncollapsed system)