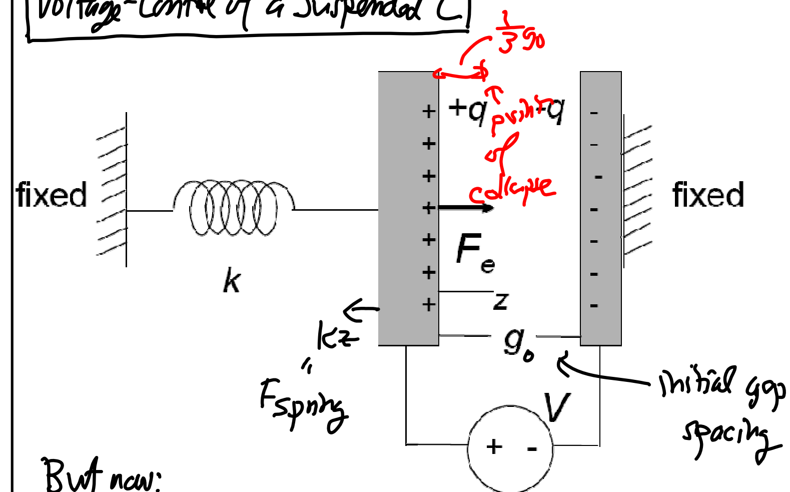
Lecture 21: Pull-in Voltage and Electrical Stiffness

- **Announcements:**
- Module 12 on Capacitive Transducers online
- HW#5 online and due Wednesday, April 13
- Project slide #1 due Friday, April 8
- ------------------------------------
- Reading: Senturia, Chpt. 5, Chpt. 6
- Lecture Topics:
  - Energy Conserving Transducers
    - Charge Control
    - Voltage Control
  - Parallel-Plate Capacitive Transducers
    - Linearizing Capacitive Actuators
    - Electrical Stiffness
  - Electrostatic Comb-Drive
    - 1st Order Analysis
    - 2nd Order Analysis
- ------------------------------------
- **Last Time:**
- In the process of determining pull-in voltage

over

Voltage-Control of a Suspended C



But now:

$$F_e = \frac{\partial W'(V,g)}{\partial g}\bigg|_q \rightarrow F_e = \frac{1}{2}\frac{\epsilon A}{g^2}V^2$$

And the gap:

$$g = g_0 - z = g_0 - \frac{F_e}{k} = \boxed{g_0 - \frac{1}{2}\frac{\epsilon A}{g^2}\frac{V^2}{k} = g}$$

g shows up on both sides!

If $V\uparrow \rightarrow g\downarrow \rightarrow F_e\uparrow$

(+) Feedback!

If loop gain > 1, then this will go unstable!

plate will collapse! (into the electrode)

Charge: (for a stable gap)

$$q = \frac{\partial W'(V,g)}{\partial V}\Big|_g = CV \checkmark \quad \text{(as expected)}$$

Stability Analysis

⇒ determine under what conditions voltage control will cause collapse of the plates:

$$F_{net} = F_e - F_{spring} = \underbrace{\frac{\epsilon A V^2}{2g^2}}_{F_e} - \underbrace{k(g_o - g)}_{spring}$$

What happens when I change g by a small increment dg?

↳ get an increment in the net attractive force F_{net}

$$\underbrace{dF_{net}}_{(-)} = \frac{\partial F_{net}}{\partial g} dg = \left[-\frac{\epsilon A V^2}{g^3} + k \right] \underbrace{dg}_{(-)}$$

If $g\downarrow \to dg = (-)$, then for stability need $F_{net}\downarrow \to dF_{net} = (-)$

This must be (+)! → otherwise, the plates collapse!

Thus: $\boxed{k > \frac{\epsilon A V^2}{g^3}}$ (for a stable uncollapsed system)

Pull-in Voltage V_{PI} & Pull-in Gap g_{PI}

$V_{PI} \triangleq$ voltage @ which plates collapse

$g_{PI} \triangleq$ gap @ " " "

The plates go unstable when:

$$k = \frac{\epsilon A V_{PI}^2}{g_{PI}^3} \qquad (1)$$

$$F_{net} = 0 = \frac{\epsilon A V_{PI}^2}{2 g_{PI}^2} - k(g_o - g_{PI}) \qquad (2)$$

Substitute (1) into (2):

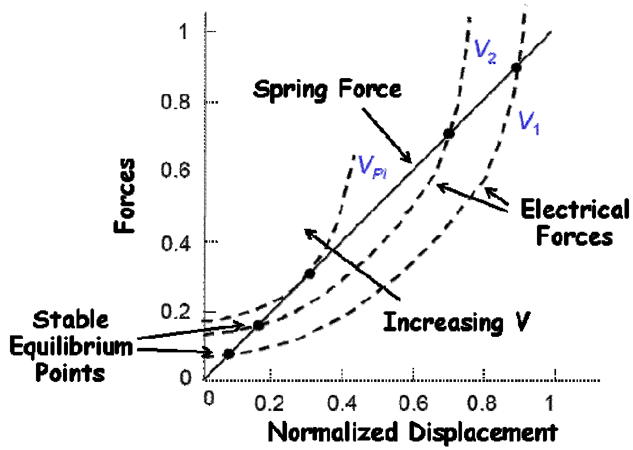
$$0 = \frac{\cancel{\epsilon A V_{PI}^2}}{2\cancel{g_{PI}^2}} - \frac{\cancel{\epsilon A V_{PI}^2}}{g_{PI}^{\cancel{3}}}(g_o - g_{PI})$$

$$\frac{g_o - g_{PI}}{g_{PI}} = \frac{1}{2} \rightarrow g_o = \frac{3}{2} g_{PI}$$

$$\therefore \boxed{g_{PI} = \frac{2}{3} g_o}$$

When the gap is driven by a voltage to (2/3) the initial gap → collapse!

$$V_{PI} = \sqrt{\frac{k g_{PI}^3}{\epsilon A}} \rightarrow \boxed{V_{PI} = \sqrt{\frac{8}{27}\frac{k g_o^3}{\epsilon A}}}$$



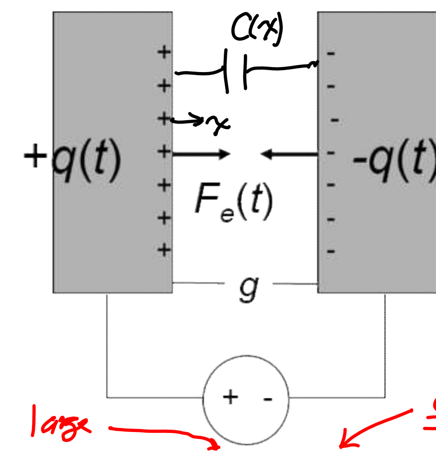
Advantages of Electrostatic Actuators:

- Easy to manufacture in micromachining processes, since conductors and air gaps are all that's needed → low cost!
- Energy conserving → only parasitic energy loss through I^2R losses in conductors and interconnects
- Variety of geometries available that allow tailoring of the relationships between voltage, force, and displacement
- Electrostatic forces can become very large when dimensions shrink → electrostatics scales well!
- Same capacitive structures can be used for both drive and sense of velocity or displacement
- Simplicity of transducer greatly reduces mechanical energy losses, allowing the highest Q's for resonant structures

Disadvantages of Electrostatic Actuators:

- Nonlinear voltage-to-force transfer function
- Relatively weak compared with other transducers (e.g., piezoelectric), but things get better as dimensions scale

$\boxed{\text{Linearizing the Voltage-to-Force Transfer Fcn.}}$



large   small

$$v(t) = \underbrace{V_P}_{\text{Bias (DC)}} + \underbrace{v_i(t)}_{\text{Signal (AC)}}$$

small-signal

$$F_e(t) = \frac{\partial W'}{\partial x} = \frac{\partial}{\partial x}\left[\frac{1}{2}C[v(t)]^2\right]$$

$$= \frac{1}{2}\frac{\partial C}{\partial x}[v(t)]^2 = \frac{1}{2}\frac{\partial C}{\partial x}[V_P + v_i(t)]^2$$

$$= \frac{1}{2}\left[V_P^2 + 2V_P v_i(t) + \cancel{[v_i(t)]^2}\right]\frac{\partial C}{\partial x}$$

$$[V_P \gg v_i(t)] \Rightarrow F_e(t) = \underbrace{\frac{1}{2}V_P^2\frac{\partial C}{\partial x}}_{\text{DC Offset}} + \underbrace{V_P\frac{\partial C}{\partial x}v_i(t)}_{\text{AC Drive Signal}}$$

$$C_o = \frac{\epsilon A}{g_o} \rightarrow C(x) = \frac{\epsilon A}{g_o - x} = C_o\left(1-\frac{x}{g_o}\right)^{-1}$$

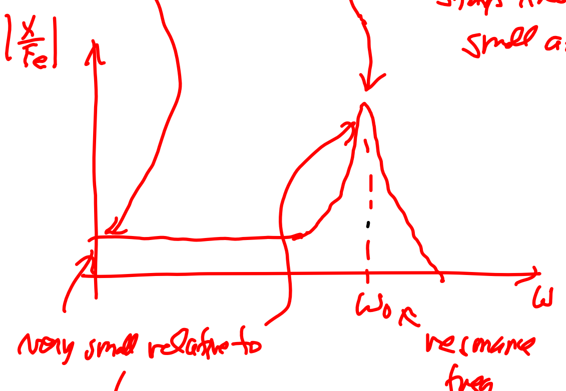
$$[x \ll g_o] \Rightarrow \approx C_o\left(1+\frac{x}{g_o}\right)$$

just taking the 1st term of Taylor series

$$\therefore \frac{\partial C}{\partial x} = \frac{C_o}{g_o} = \frac{\epsilon A}{g_o^2}$$

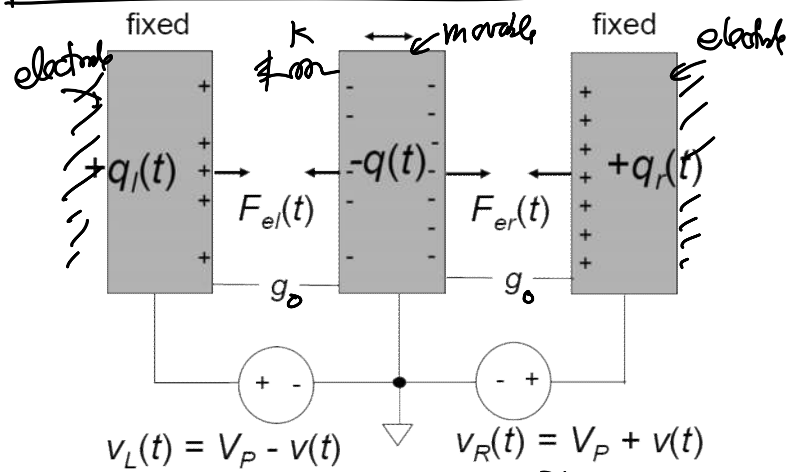
$$\Rightarrow F_e(t) = \underbrace{\frac{1}{2}\frac{C_o}{g_o}V_P^2}_{\text{DC Offset}} + \underbrace{V_P\frac{C_o}{g_o}v_i(t)}_{\text{AC component}}$$

$\sim V_P$ allows large gain $v_i \rightarrow F_e$ → stays linear for small amplitudes



But still must worry about V_{PI} → pull-in

Cancel the DC Offset Using Differential Symmetry



differential input into balanced structure

$$F_{net} = F_{er}(t) - F_{el}(t)$$

$$= \frac{1}{2}\frac{\partial C}{\partial x}\left\{[v_R(t)]^2 - [v_L(t)]^2\right\}$$

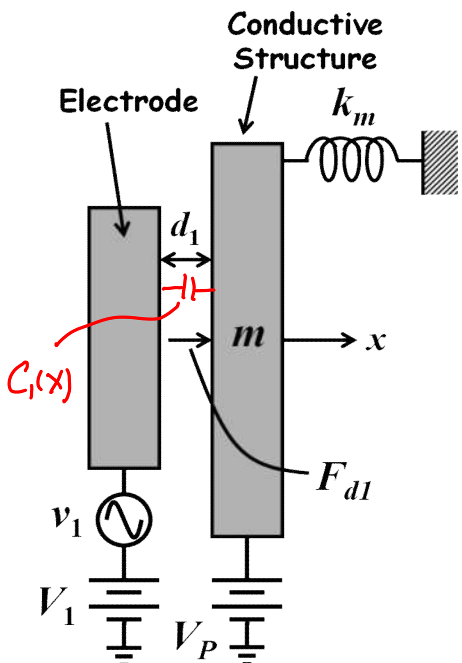
$$= \frac{1}{2}\frac{\partial C}{\partial x}\left\{\cancel{V_P^2} + 2V_P v(t) + \cancel{[v(t)]^2} - \left(\cancel{V_P^2} - 2V_P v(t) + \cancel{[v(t)]^2}\right)\right\}$$

$$\therefore \boxed{F_{net}(t) = 2V_P\frac{\partial C}{\partial x}v(t) = 2V_P\frac{C_o}{g_o}v(t)}$$

→ No DC component → less pull-in problems, but only to the extent of matching

→ Linear w/ $v(t)$!

$\frac{\partial C}{\partial x}$ Nonlinearity Still Affects Us!



More Complete $\frac{\partial C}{\partial x}$ Expression

$$C_1(x) = \frac{\epsilon A}{d_1+x} = C_{o1}\left(1+\frac{x}{d_1}\right)^{-1} \rightarrow \frac{\partial C}{\partial x} = -\frac{C_{o1}}{d_1}\left(1+\frac{x}{d_1}\right)^{-2}$$

[Expand into Taylor Series]

$$\frac{\partial C_1}{\partial x} = -\frac{C_{o1}}{d_1}\left(1+A_1x+A_2x^2+A_3x^3+\cdots\right)$$

where $A_1 = -\frac{2}{d_1}$, $A_2 = \frac{3}{d_1^2}$, $A_3 = -\frac{4}{d_1^3}$, ...

$$F_{d1} = \frac{1}{2}\frac{\partial C_1}{\partial x}(V_P - V_1 - v_1)^2 = \frac{1}{2}\frac{\partial C}{\partial x}(V_{P1} - v_1)^2$$

$V_{P1} = V_P - V_1$

$\frac{|v_1|^2}{2}(1+\cos 2\omega_o t)$

$= v_1^2 = |v_1|^2\cos^2\omega_o t$

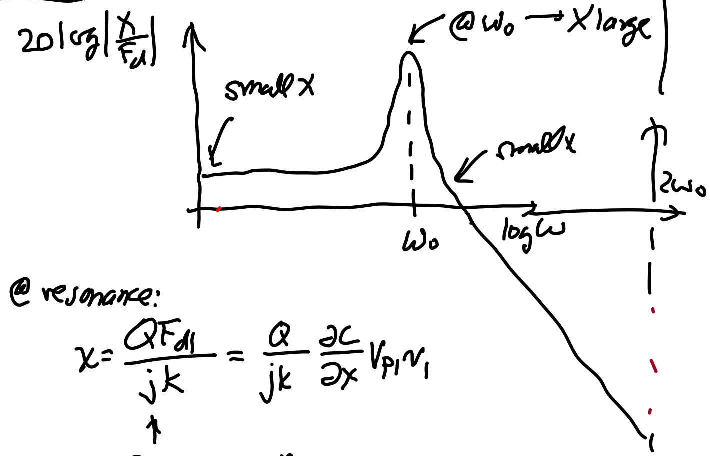
[Small displacements: $x \ll d_1$]

$$F_{d1} = \frac{1}{2}\left(-\frac{C_{o1}}{d_1}\right)(1+A_1x)\left(V_{P1}^2 - 2V_{P1}v_1 + v_1^2\right)$$

$v_1 = |v_1|\cos\omega_o t$

$$= \frac{1}{2}\left(-\frac{C_{o1}}{d_1}\right)\left\{V_{P1}^2 - 2V_{P1}v_1 + v_1^2 + A_1V_{P1}^2x - 2A_1V_{P1}xv_1 + A_1xv_1^2\right\}$$

Resonance:



@ resonance:

$$x = \frac{QF_{d1}}{jk} = \frac{Q}{jk}\frac{\partial C}{\partial x}V_{P1}v_1$$

90° phase shift

$v_1 = |v_1|\cos\omega_o t \rightarrow x = |x|\sin\omega_o t$

90° phase-shift

Force Terms @ ω_o

$$F_{d1}\big|_{\omega_o} = \underbrace{V_{P1}\frac{C_{o1}}{d_1}|v_i|\cos\omega_o t}_{\text{drive force term}} + V_{P1}^2\frac{C_{o1}}{d_1^2}|x|\sin\omega_o t$$

k_e = electrical stiffness

proportional to x

90° phase-shifted

∴ in phase w/ displacement!

∴ it's a stiffness!

Electrical Stiffness:

① A negative spring constant.

② Derives from V_P:

$$\boxed{k_e = V_{P1}^2\frac{C_{o1}}{d_1^2} = V_{P1}^2\frac{\epsilon A}{d_1^3}}$$

overlap area of C

DC Bias

3rd power dependence on gap!

$k_e \rightarrow$ can affect resonance freq, f_o

$\omega_o \triangleq$ radian resonance freq. w/ no V_P applied (i.e., $V_{P1}=0V$)

$\rightarrow \omega_o = \sqrt{\frac{k_m}{m}}$

$\omega_o' = \sqrt{\frac{k}{m}} = \sqrt{\frac{k_m - k_e}{m}}$ } w/ V_P = finite

* $= \sqrt{\frac{k_m}{m}}\left(1 - \frac{k_e}{k_m}\right)^{1/2}$

$$\boxed{\omega_o' = \omega_o\left[1 - \frac{V_{P1}^2}{k_m}\frac{\epsilon A}{d_1^3}\right]^{1/2}}$$

Now, a function of DC-Bias! (voltage-controllable!)

- Go through Module 12 slides 26-35