Lecture Outline

- Reading: Senturia, Chpt. 5, Chpt. 6
- Lecture Topics:
  - Energy Conserving Transducers
    - Charge Control
    - Voltage Control
  - Parallel-Plate Capacitive Transducers
    - Linearizing Capacitive Actuators
    - Electrical Stiffness
  - Electrostatic Comb-Drive
    - 1st Order Analysis
    - 2nd Order Analysis
Basic Physics of Electrostatic Actuation

• Goal: Determine gap spacing \( g \) as a function of input variables
• First, need to determine the energy of the system
• Two ways to change the energy:
  - Change the charge \( q \)
  - Change the separation \( g \)

\[
\Delta W(q,g) = V\Delta q + F_e \Delta g
\]
\[
dW = Vdq + F_e dg
\]

• Note: We assume that the plates are supported elastically, so they don't collapse

Stored Energy

• Here, the stored energy is the work done in increasing the gap after charging capacitor at zero gap

\[
W = 0 + \int_0^g F_e dg' = \frac{1}{2} \varepsilon \varepsilon_0 \frac{q^2}{g^2}
\]

\[\text{Cross-sectional Area}\]

\[
q' \rightarrow W(q,g)
\]

\[
\text{zero gap} \rightarrow \text{zero stored energy}
\]

For a capacitor \( C \):

\[
q = CV \rightarrow V = \frac{q}{C}
\]

\[
W(q) = \int_0^q V dq = \int_0^q \left( \frac{1}{2} \varepsilon \varepsilon_0 \frac{q^2}{g^2} \right) dq
\]

Work done to change \( C \) to \( q \) at fixed gap:

\[
\frac{1}{2} \varepsilon \varepsilon_0 \frac{q^3}{g^2} = \frac{1}{2} \varepsilon \varepsilon_0 \frac{q^3}{g^2} \cdot W(q)
\]
**Charge-Control Case**

- Having found stored energy, we can now find the force acting on the plates and the voltage across them:

  \[ W(q,g) = \frac{1}{2} \varepsilon_0 \frac{q^2}{g} \]

  \[ F_e = -\frac{\partial W(q,g)}{\partial g} = \frac{1}{2} \varepsilon_0 \frac{q^2}{g^2} \]

  \[ V = -\frac{q}{C} \]

  \[ \Rightarrow \text{force is given by:} \]

  \[ F_e = \frac{1}{2} \varepsilon_0 \frac{q^2}{g^2} \quad \text{and} \quad V = -\frac{q}{C} \]

  \[ \Rightarrow \text{voltage is given by:} \]

  \[ V = -\frac{1}{2} \varepsilon_0 \frac{q^2}{g^2} \]

  \[ \Rightarrow \text{force is independent of gap spacing!} \]

**Voltage-Control Case**

- Practical situation: We control \( V \)
  - Charge control on the typical sub-pF MEMS actuation capacitor is difficult
  - Need to find \( F_e \) as a partial derivative of the stored energy \( W = W(V,g) \) with respect to \( g \) with \( V \) held constant? But can't do this with present \( W(q,g) \) formula
  - Solution: Apply Legendre transformation and define the co-energy \( W'(V,g) \)

  \[ W'(V,g) = W(V,g) - \Phi(q) \]

  \[ W'(V,e) = \int_0^q \Phi(q) dq \]

  \[ \Rightarrow \text{can define co-energy as:} \]

  \[ W'(e) = eq - W(q) \]

  \[ \text{(from this plot)} \]
Co-Energy Formulation

**Co-Energy Formulation**

- For our present problem (i.e., movable capacitive plates), the co-energy formulation becomes

\[ W'(V, g) = qV - W(q, g) \]

Differentially, this becomes:

\[ \frac{dW'(V, g)}{dV} = (qgV + Vdq) - \frac{dW(q, g)}{dq} \]

But \[ \frac{dW(q, g)}{dq} = Fedg + Vdq \]

\[ \frac{dW'(V, g)}{g} = qgV - Fedg \]

From which:

- Charge, \( Q = \frac{dW'(V, g)}{dV} \mid_{g \text{ const.}} \)
- Force, \( F_e = -\frac{dW'(V, g)}{dg} \mid_{V \text{ const.}} = \frac{dW'(V, g)}{dV} \mid_{V \text{ const.}} \)

Electrostatic Force (Voltage Control)

- Find co-energy in terms of voltage (with gap held constant)

\[ W' = \int_{0}^{V} q(g, V')dV' = \int_{0}^{V} \left( \frac{A}{g} \right) V'dV' = \frac{1}{2} \left( \frac{\varepsilon A}{g} \right) V^2 = \frac{1}{2} CV^2 \]

(as expected)

- Variation of co-energy with respect to gap yields electrostatic force:

\[ F_e = -\frac{\partial W'(V, g)}{dg} \mid_{V} = -\frac{1}{2} \left( \frac{\varepsilon A}{g^2} \right) V^2 = \frac{1}{2} \frac{C}{g} V^2 \]

strong function of gap!

- Variation of co-energy with respect to voltage yields charge:

\[ q = \frac{\partial W'(V, g)}{dV} \mid_{g} = \left( \frac{\varepsilon A}{g} \right) V = CV \]

as expected
Spring-Suspended Capacitive Plate

Charge Control of a Spring-Suspended Capacitive Plate

Force generated by charge \( q \) applied by current \( I \):

\[
F_e = \frac{\partial V(q, q)}{\partial q} = \frac{q^2}{2eA}
\]

Restoring force of spring: \( F_{spring} = k z = F_e \) (at equilibrium)

And the gap:

\[
g = g_0 - z = g_0 - \frac{F_e}{k} = \frac{1 - \frac{d^2}{2eA}}{k} \Rightarrow \text{Can increase } q \text{ and drive } \quad g \rightarrow 0
\]

Initial gap

\[
V = \frac{q}{C} = \frac{q}{eA} g = \frac{q}{eA} \left( g_0 - \frac{1}{2eA} \frac{1}{k} \right) = V \Rightarrow \text{W as } g = \frac{1}{eA} \left( 1 - \frac{d^2}{2eA} \right)
\]
Voltage Control of a Spring-Suspended C

Again, $F_{\text{spring}} = kz = F_e$

But now:

$$F_e = \frac{\partial W(V,g)}{\partial g} = g \frac{\epsilon A V^2}{2 g^2}$$

And the gap:

$$g = g_0 - \frac{F_e}{k} = \sqrt{g_0 - \frac{\epsilon A V^2}{g^2 k} g}$$

Charge: (for a stable gap)

$$q = \frac{\partial W(V,g)}{\partial V} = CV = \frac{\epsilon A V^2}{g}$$

Feedback!

If loop gain $\gg 1$, then this will go unstable!

Stability Analysis

• Net attractive force on the plate:

$$F_{\text{net}} = \frac{\epsilon A V^2}{2 g^2} - k (g_0 - g)$$

• An increment in gap $dg$ leads to an increment in net attractive force $dF_{\text{net}}$

$$dF_{\text{net}} = \frac{\partial F_{\text{net}}}{\partial g} dg = -k (g_0 - g) dg$$

Thus, need

$$k > \frac{\epsilon A V^2}{g^2}$$

(for a stable uncollapsed state)
**Pull-In Voltage \( V_{PI} \)**

- \( V_{PI} \) = voltage at which the plates collapse
- The plate goes unstable when

\[
\frac{k}{g_{PI}^3} = \frac{\varepsilon A V_{PI}^2}{2 g_{PE}^2} = -\frac{\varepsilon A V_{PI}^2}{2 g_{PI}^2} = \frac{k (g_o - g_{PI})}{g_{PI}}
\]

Substituting (1) into (2):

\[
0 = \frac{\varepsilon A V_{PI}^2}{2 g_{PI}^2} - \frac{\varepsilon A V_{PI}^2}{g_{PE}^2} (g_o - g_{PI})
\]

\[
V_{PI} = \frac{k g_{PE}^2}{\varepsilon A}
\]

\[
g_{PI} = \frac{2}{3} g_o
\]

When a gap is driven by a voltage to \((2/3)\) its original spacing, collapse will occur!

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**Voltage-Controlled Plate Stability Graph**

- Below: Plot of normalized electrostatic and spring forces vs. normalized displacement \((g/g_o)\)

[Graph showing forces vs. normalized displacement]
Advantages of Electrostatic Actuators

- Easy to manufacture in micromachining processes, since conductors and air gaps are all that's needed → low cost!
- Energy conserving → only parasitic energy loss through $I^2R$ losses in conductors and interconnects
- Variety of geometries available that allow tailoring of the relationships between voltage, force, and displacement
- Electrostatic forces can become very large when dimensions shrink → electrostatics scales well!
- Same capacitive structures can be used for both drive and sense of velocity or displacement
- Simplicity of transducer greatly reduces mechanical energy losses, allowing the highest Q’s for resonant structures

Problems With Electrostatic Actuators

- Nonlinear voltage-to-force transfer function
- Relatively weak compared with other transducers (e.g., piezoelectric), but things get better as dimensions scale
Linearizing the Voltage-to-Force Transfer Function

- Apply a DC bias (or polarization) voltage $V_p$ together with the intended input (or drive) voltage $v_i(t)$, where $V_p \gg v_i(t)$

$$v(t) = V_p + v_i(t)$$

$$F_e(t) \propto \frac{\partial W}{\partial x} \cdot \frac{2}{x} \left(\frac{1}{2} C \left[\frac{1}{W} \frac{\partial W}{\partial x}\right]^2\right)$$

$$= \frac{1}{2} \frac{2C}{x} \left[\frac{1}{W} \frac{\partial W}{\partial x}\right]^3 = \frac{1}{2} (V_p + \alpha x(t))^2 \frac{2C}{x}$$

$$= \frac{1}{2} \left[V_p^2 + 2V_p \alpha x(t) + \left[\alpha x(t)\right]^2\right] \frac{2C}{x}$$

$[V_p \gg \alpha x(t)] \Rightarrow F_e(t) \approx \frac{1}{2} V_p^2 \frac{\alpha C}{x} + V_p \frac{\alpha C}{x} \alpha x(t)$

DC Offset AC drive signal

$$C(x) = \frac{\varepsilon A}{g \cdot x} \approx C_o \left(1 - \frac{x}{g_o}\right)^{-1} \approx C_o \left(1 + \frac{x}{g_o}\right)$$

$[x \ll g_o]$ ~ constant ~ linear

$$\Rightarrow F_e(t) \approx \frac{1}{2} \frac{\alpha C}{g_o} V_p^2 + V_p \frac{\alpha C}{g_o} \alpha x(t)$$
Differential Capacitive Transducer

- The net force on the suspended center electrode is:
  
  \[ F_{net} = F_{e_r}(t) - F_{e_l}(t) \]

Do the math.

Assume matched gaps.

\[ F_{net}(t) = \frac{1}{2} \frac{dC}{dx} \left\{ [V_L(t)]^2 - [V_R(t)]^2 \right\} \]

\[ = \frac{1}{2} \frac{dC}{dx} \left\{ \left(V_p^2 + 2V_pN(t) + [N(t)]^2\right) - \left(V_p^2 - 2V_pN(t) + [N(t)]^2\right) \right\} \]

\[ = \frac{1}{2} \frac{dC}{dx} \left\{ 2V_pN(t) \right\} \frac{dC}{dx} \left\{ N(t) \right\} \]

\[ \Rightarrow \text{Linear if } N(t) \] (gap match limited)

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Remaining Nonlinearity

(Electrical Stiffness)
Parallel-Plate Capacitive Nonlinearity

- **Example**: clamped-clamped laterally driven beam with balanced electrodes

- **Nomenclature**:
  - $V_a$ or $V_A$:
  - $v_a = |v_a| \cos \omega t$
  - $V_a$ or $V_A = V_A + v_a$

  **Nomenclature**:
  - **Total Value**
    - DC Component (upper case variable; upper case subscript)
    - AC or Signal Component (lower case variable; lower case subscript)

**Parallel-Plate Capacitive Nonlinearity**

- **Example**: clamped-clamped laterally driven beam with balanced electrodes

- **Expression for $\partial C/\partial x$**:
  - $C_i(x) \cdot \frac{A}{d_i + x} : \frac{C_{G1} (1 + \frac{x}{d_i})^{-1}}{d_i} \rightarrow \frac{\partial C_{G1}}{\partial x} = \frac{C_{G1}}{d_i} \left(1 + \frac{x}{d_i}\right)^{-2}$

  **Expand the Taylor series further**
  - $\frac{C_{G1}}{d_i} \left(1 + A_1 x + A_2 x^2 + A_3 x^3 + \cdots \right)$
  - Where:
    - $A_1 = -\frac{2}{d_i}$
    - $A_2 = \frac{3}{d_i^2}$
    - $A_3 = -\frac{4}{d_i^3}$
    - $\vdots$
Parallel-Plate Capacitive Nonlinearity

Thus, the expression for force from the left side becomes:

\[ F_{dl} = \frac{1}{2} \frac{\partial}{\partial x} (V_p - V_i - N_i)^2 = \frac{1}{2} \frac{\partial}{\partial x} (V_p - V_i)^2 \]

\[ \text{small displacements : } x \ll d_i \]

\[ F_{dl} = \frac{1}{2} \left( \frac{C_{ol}}{d_i} \right) \left( (V_i - 2V_p)N_i + \omega N_i^2 \right) \]

\[ = \frac{1}{2} \left( \frac{C_{ol}}{d_i} \right) \left( V_i^2 - 2V_pN_i + N_i^2 \right) \]

@ resonance: \( x = \frac{\partial F_{dl}}{\partial V_i} = \frac{\partial}{\partial V_p} V_i \cdot V_i \)

Thus:

\[ N_i = \text{input to } \epsilon \rightarrow x = \text{at } 90^\circ \text{ phase shifted from } N_i \]

Drive force arising from the input excitation voltage at the frequency of this voltage is proportional to displacement.

90° phase-shifted from drive, so in phase with displacement.

These two together mean that this force acts against the spring restoring force!

A negative spring constant

Since it derives from \( V_p \), we call it the electrical stiffness, given by:

\[ k_e = V_p^2 \frac{C_{ol}}{d_i^2} = V_p^2 \frac{EA}{d_i^2} \]
**Electrical Stiffness, \( k_e \)**

- The electrical stiffness \( k_e \) behaves like any other stiffness.
- It affects resonance frequency:

\[
\omega'_o = \omega_o \left(1 - \frac{V_{P1}^2 \varepsilon A}{k_m d_1^3}\right)^{1/2}
\]

Frequency is now a function of dc-bias \( V_{P1} \).

**Voltage-Controllable Center Frequency**

- Quadrature force voltage-controllable electrical stiffness:

\[
k_e = \frac{\varepsilon_o A}{d^3} V_P^2
\]

\[
f_o = \frac{1}{2\pi} \sqrt{\frac{k_m - k_e}{m}}
\]
Microresonator Thermal Stability

- Thermal stability of poly-Si micromechanical resonator is 10X worse than the worst case of AT-cut quartz crystal at various angles.

Geometric-Stress Compensation

- Use a temperature dependent mechanical stiffness to null frequency shifts due to Young's modulus thermal dep.

**Problems:**
- Stress relaxation
- Compromised design flexibility

[Hsu et al, IEDM 2000]
Voltage-Controllable Center Frequency

- Quadrature force \( k_e \) voltage-controllable electrical stiffness:
  \[ k_e = \frac{\varepsilon A_o V^2}{d^3} \]
  \[ f_o = \frac{1}{2\pi} \sqrt{\frac{k_m - k_e}{m_r}} \]

Excellent Temperature Stability

- Top Electrode-to-Resonator Gap \( \uparrow \)
- Elect. Stiffness: \( k_e \sim 1/d^2 \) \( \uparrow \)
- Frequency: \( f_o \sim (k_m - k_e)^{0.5} \) \( \uparrow \)
- Counteracts reduction in frequency due to Young's modulus temp. dependence

Elect.-Stiffness Compensation
- On par with quartz!
- AT-cut Quartz Crystal at Various Cut Angles
- Uncompensated resonator

- Volterra Mass
- Uncompensated resonator

[Ref: Hafner]

[Electro. Stiffness: \( k_e \sim 1/d^3 \)]

[Ref: Hsu et al MEMS'02]
**Measured Δf/ΔT vs. T for ke-compensated μResonators**

- Slits help to release the stress generated by lateral thermal expansion
- Linear TC curves
- -0.24ppm/°C

**Design/Performance:**
- $f_0 = 10$ MHz, $Q = 4,000$
- $V_P = 8$ V, $h = 4$ μm
- $d_o = 1000$ Å, $h = 2$ μm
- $W = 8$ μm, $L = 40$ μm

[Hsu et al MEMS'02]

Can One Cancel $k_e$ with Two Electrodes?

- What if we don’t like the dependence of frequency on $V_P$?
- Can we cancel $k_e$ via a differential input electrode configuration?
- If we do a similar analysis for $F_{d2}$ at Electrode 2:

\[
F_{d2}|_{\omega_o} = -V_{P2} \frac{C_{\omega_o}}{d_2} |v_2| \cos \omega_o t + V_{P2}^2 \frac{C_{\omega_o}}{d_2^2} |x| \sin \omega_o t
\]

- Subtracts from the $F_{d1}$ term, as expected
- Adds to the quadrature term $\rightarrow k_e$'s add, no matter the electrode configuration!
Problems With Parallel-Plate C Drive

* Nonlinear voltage-to-force transfer function
  - Resonance frequency becomes dependent on parameters (e.g., bias voltage $V_p$)
  - Output current will also take on nonlinear characteristics as amplitude grows (i.e., as $x$ approaches $d_o$)
  - Noise can alias due to nonlinearity

* Range of motion is small
  - For larger motion, need larger gap ... but larger gap weakens the electrostatic force
  - Large motion is often needed (e.g., by gyroscopes, vibromotors, optical MEMS)

Electrostatic Comb Drive
Electrostatic Comb Drive

- Use of comb-capacitive transducers brings many benefits
  - Linearizes voltage-generated input forces
  - (Ideally) eliminates dependence of frequency on dc-bias
  - Allows a large range of motion

Comb-Driven Folded Beam Actuator

Comb-Drive Force Equation (1st Pass)

\[ F_d = \frac{2h^2}{d^2} \left( \frac{\partial^2 x}{\partial x^2} \right) \left( \frac{\partial^2 x}{\partial t^2} - 2V_p \nu \frac{\partial x}{\partial t} + \nu x^2 \right) \approx -2V_p \frac{\partial^2 x}{\partial t^2} \nu \frac{x^2}{d^2} = F_d \]

But wait! This ignores other practical effects! (No dependence on \( x \)!) (LINEAR!)
Lateral Comb-Drive Electrical Stiffness

\[ C(x) = \frac{2N\varepsilon h}{d} \frac{\partial C}{\partial x} = \frac{2N\varepsilon h}{d} \]

- Again: No \((\partial C/\partial x) \times\)-dependence \(\rightarrow\) no electrical stiffness: \(k_e = 0\)
- Frequency immune to changes in \(V_P\) or gap spacing!

Typical Drive & Sense Configuration

\[ F_{d1} = \frac{1}{2} \frac{\partial}{\partial x} (N_z V_P) = \frac{1}{2} \left( \varepsilon \frac{h}{d} \right) (N_z V_P) \]
\[ F_{d2} = \frac{1}{2} \frac{\partial}{\partial x} (N_z V_P) = \frac{1}{2} \left( \varepsilon \frac{h}{d} \right) (N_z V_P) \]
\[ F_{net} = \frac{1}{2} \varepsilon \frac{h}{d} (N_z V_P) \]

\[ \text{Simple Analysis:} \]
\[ F_{d1} + F_{d2} = \frac{1}{2} \varepsilon \frac{h}{d} (N_z V_P)^2 - 2(N_e V_e V_i + V_i^2)(2N_z) \]
Comb-Drive Force Equation (2nd Pass)

- In our 1st pass, we accounted for
  - Parallel-plate capacitance between stator and rotor
- ... but neglected:
  - Fringing fields
  - Capacitance to the substrate
- All of these capacitors must be included when evaluating the energy expression!

\[ F_{e,x} = \frac{\partial W'}{\partial x} = \frac{1}{2} \frac{dC_{sp}}{dx} V_s^2 + \frac{1}{2} \frac{dC_{rp}}{dx} V_r^2 + \frac{1}{2} \frac{dC_{rs}}{dx} (V_s - V_r)^2 \]

Comb-Drive Force With Ground Plane Correction

- Finger displacement changes not only the capacitance between stator and rotor, but also between these structures and the ground plane → modifies the capacitive energy

\[ F_{e,x} = \frac{\partial W'}{\partial x} = \frac{1}{2} \frac{dC_{sp}}{dx} V_s^2 + \frac{1}{2} \frac{dC_{rp}}{dx} V_r^2 + \frac{1}{2} \frac{dC_{rs}}{dx} (V_s - V_r)^2 \]

[Gary Fedder, Ph.D., UC Berkeley, 1994]
Capacitance Expressions

- Case: \( V_r = V_p = 0 \) V
- \( C_{sp} \) depends on whether or not fingers are engaged

\[
C_{sp} = N[C'_{sp,e}x + C'_{sp,m}(L - x)]
\]

\[
C_{rs} = NC'_{rs}x
\]

Region 2

Region 3

[Gary Fedder, Ph.D., UC Berkeley, 1994]

Comb-Drive Force With Ground Plane Correction

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\[
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\]

\[
F_{e,x} = \frac{N}{2} (C'_{rs} + C'_{sp,e} - C'_{sp,m})V_s^2
\]

(for \( V_r = V_p = 0 \))

[Gary Fedder, Ph.D., UC Berkeley, 1994]
Simulate to Get Capacitors → Force

* Below: 2D finite element simulation

\[ F_{e,x} = \frac{N}{2} \left( C_{rs} + C_{sp,e} - C_{sp,s} \right) V^2 \]

20-40% reduction of \( F_{e,x} \)

Vertical Force (Levitation)

\[ F_{e,z} = \frac{\partial W'}{\partial z} = \frac{1}{2} \frac{dC_{sp}}{dz} V_s^2 + \frac{1}{2} \frac{dC_{tp}}{dz} V_r^2 + \frac{1}{2} \frac{dC_{rs}}{dz} (V_s - V_r)^2 \]

* For \( V_r = 0 \text{V} \) (as shown):

\[ F_{e,z} = \frac{1}{2} N x \left[ \frac{d(C_{sp,e}' + C_{rs}')}{dz} \right] V_s^2 \]
**Simulated Levitation Force**

* Below: simulated vertical force $F_z$ vs. $z$ at different $V_p$'s [f/ Bill Tang Ph.D., UCB, 1990]

See that $F_z$ is roughly proportional to $-z$ for $z$ less than $z_o$ → it's like an electrical stiffness that adds to the mechanical stiffness

$$F_z \approx \gamma z V_p^2 \frac{(z_o - z)}{z_o} = k_e(z_o - z)$$

**Vertical Resonance Frequency**

Vertical resonance frequency

$$\omega_z = \sqrt{\frac{k_z + k_e}{k_z}}$$

where

$$k_e = \left( \frac{\gamma z}{z_o} \right) V_p^2$$

* Signs of electrical stiffnesses in MEMS:
  - Comb (x-axis) → $k_e = 0$
  - Comb (z-axis) → $k_e > 0$
  - Parallel Plate → $k_e < 0$
Suppressing Levitation

- Pattern ground plane polysilicon into differentially excited electrodes to minimize field lines terminating on top of comb
- **Penalty**: x-axis force is reduced

Force of Comb-Drive vs. Parallel-Plate

- **Comb drive** (x-direction)
  \[ F_{e,x} = \frac{1}{2} \frac{\varepsilon_0 h L_f}{d_o^2} V_s^2 \]

- **Differential Parallel-Plate**
  (y-direction)
  \[ F_{e,y} = \frac{1}{2} \frac{\varepsilon_0 h L_d}{d_o^2} V_2^2 \]

Gap = \( d_o = 1 \ \mu m \)
Thickness = \( h = 2 \ \mu m \)
Finger Length = \( L_f = 100 \ \mu m \)
Finger Overlap = \( L_d = 75 \ \mu m \)

Parallel-plate generates a much larger force; but at the cost of linearity