EE C247B – ME C218
Introduction to MEMS Design
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Lecture Module 8: Microstructural Elements

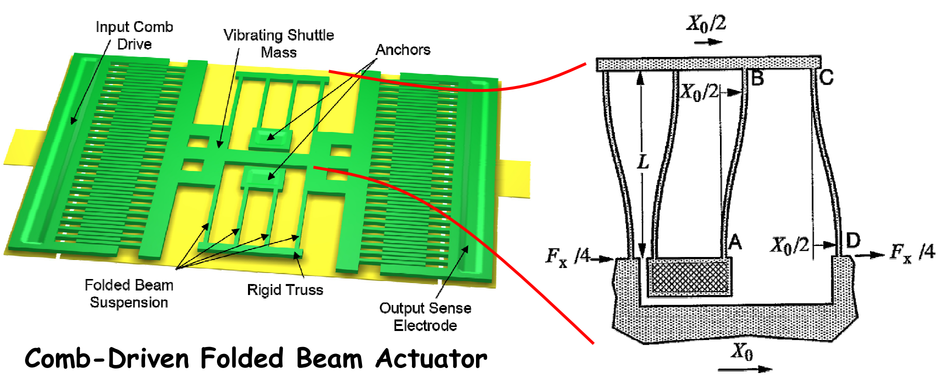
Outline

- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients

Bending of Beams

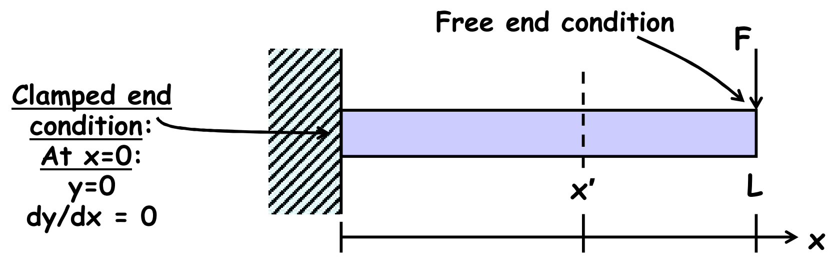
Beams: The Springs of Most MEMS

- Springs and suspensions very common in MEMS
  - Coils are popular in the macro-world; but not easy to make in the micro-world
  - Beams: simpler to fabricate and analyze; become "stronger" on the micro-scale → use beams for MEMS



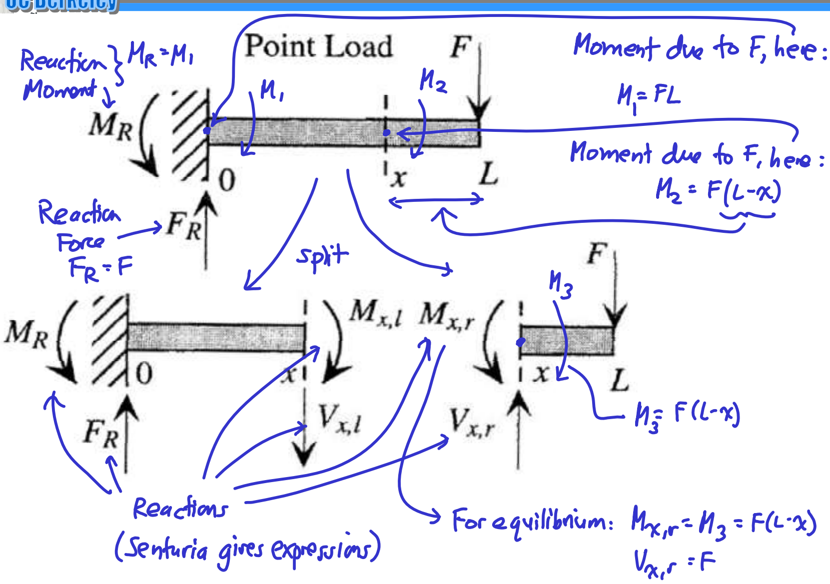
Comb-Driven Folded Beam Actuator

Bending a Cantilever Beam

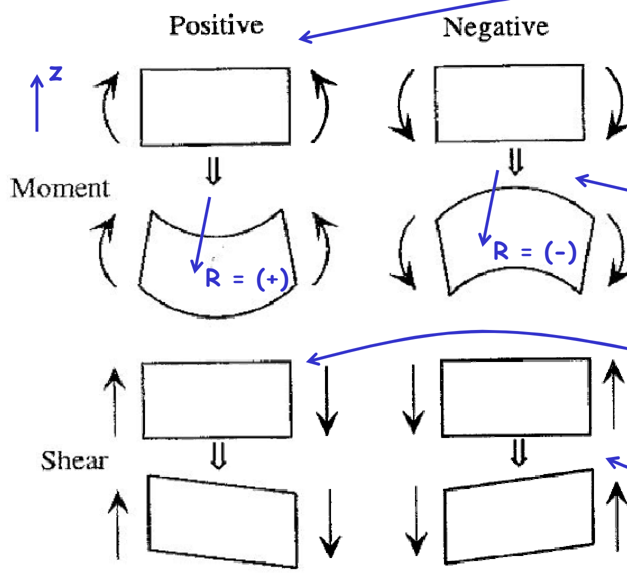


- Objective: Find relation between tip deflection $y(x=L_c)$ and applied load F
- Assumptions:
  1. Tip deflection is small compared with beam length
  2. Plane sections (normal to beam's axis) remain plane and normal during bending, i.e., "pure bending"
  3. Shear stresses are negligible

Reaction Forces and Moments



Sign Conventions for Moments & Shear Forces



(+) moment leads to deformation with a (+) radius of curvature (i.e., upwards)

(-) moment leads to deformation with a (-) radius of curvature (i.e., downwards)

(+) shear forces produce clockwise rotation

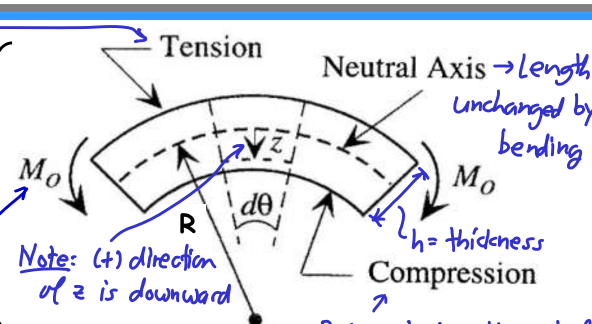
(-) shear forces produce counter-clockwise rotation

Beam Segment in Pure Bending

Portions above the neutral axis go into

Small section of a beam bent in response to a tranverse load

Applied Moment



→ Length unchanged by bending

h = thickness

Note: (+) direction of z is downward

Portions below the neutral axis go into compression

Consider a segment bounded by the dashed lines defined by $d\theta$:

At $z=0$: (i.e, at the neutral axis): segment length $= dx = R\,d\theta$ (1)

At any z: segment length $= dL = (R-z)\,d\theta$ (2)

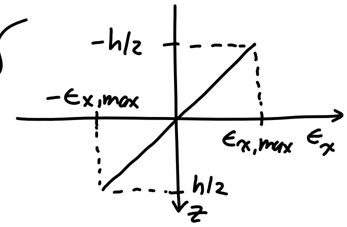
Combining (1) & (2): $dL = dx - z\,d\theta = dx - \frac{z}{R}dx$

Beam Segment in Pure Bending (cont.)

Thus, the axial strain @ z: $\epsilon_x = \frac{dL - dx}{dx} = -\frac{z}{R} \Rightarrow \boxed{\epsilon_x = -\frac{z}{R}}$

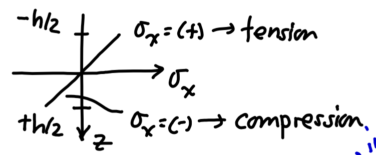
[dx = original (unstressed) segment length]

Thus, strain varies linearly along beam thickness, and has a maximum value $\epsilon_{x,max} = \frac{h/2}{R}$



Of course, there is a corresponding axial stress:

$$\sigma_x = \epsilon_x E = \boxed{-\frac{zE}{R} = \sigma_x}$$

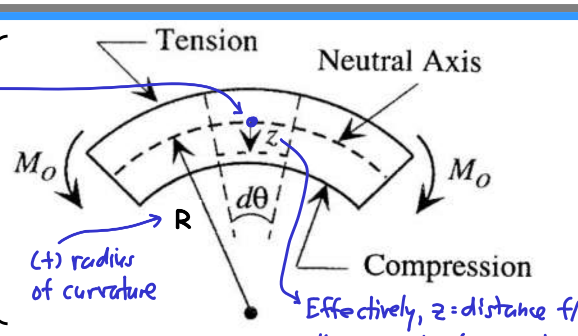


This gradient in stress then generates a bending moment... in response to the applied moment!

Internal Bending Moment

Moment around this point

Small section of a beam bent in response to a transverse load



(+) radius of curvature

Effectively, z = distance f/ the moment reference pt.

To get the bending moment:

$\Rightarrow$ integrate stress through the thickness of the beam

$$M = \int_{-h/2}^{h/2} [\underbrace{(W dz)\sigma_x}_{\text{force}}] \cdot z = -\int_{-h/2}^{h/2} \frac{EWz^2}{R} dz \Rightarrow \boxed{M = -\left(\frac{1}{12} W h^3\right)\frac{E}{R}}$$

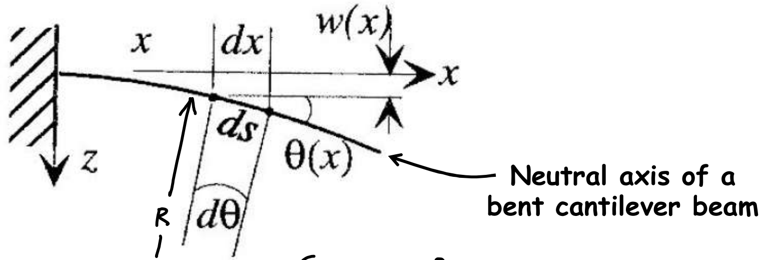
$\left[\sigma_x = -\frac{zE}{R}\right]$

$\frac{1}{12} W h^3 = I \triangleq$ Moment of Inertia

$$\boxed{\frac{1}{R} = -\frac{M}{EI}}$$

Note: (+) radius of curvature $\rightarrow$ (-) internal bending moment!

Differential Beam Bending Equation



Write out geometric relationships:

[Small Angle Approx.]

$\cos\theta = \frac{dx}{ds} \rightarrow ds = \frac{dx}{\cos\theta} \longrightarrow ds \approx dx$

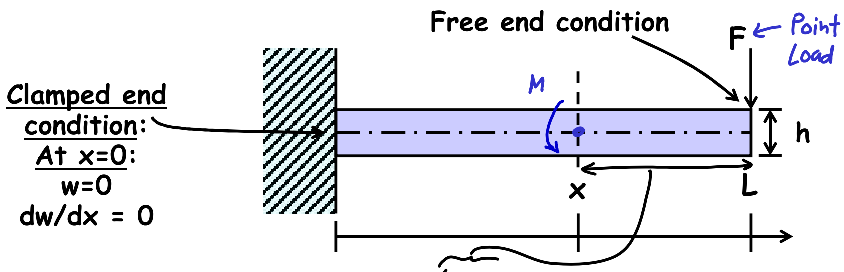
$\tan\theta = \frac{dw}{dx}$ = slope of beam @ any point $\longrightarrow \theta \approx \frac{dw}{dx}$ (1)

$ds = R\,d\theta \rightarrow \frac{1}{R} = \frac{d\theta}{ds} \longrightarrow \frac{1}{R} = \frac{d\theta}{dx}$ (2)

Inserting (1) in (2): $\frac{1}{R} = \boxed{\frac{d^2w}{dx^2} = -\frac{M}{EI}}$ [Differential Equation for Small Angle Bending of Beams]

Example: Cantilever Beam w/ a Concentrated Load

Cantilever Beam w/ a Concentrated Load



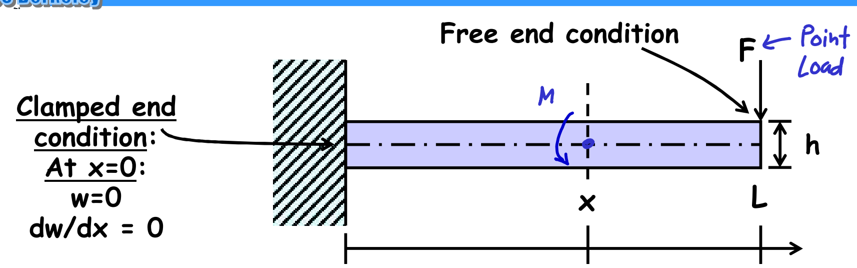
Internal Moment @ position x: $M = -F(L-x)$

Thus: $\frac{d^2w}{dx^2} = \frac{F}{EI}(L-x)$, w/ $\begin{cases} \text{Clamped End B.C.'s: } w(x=0)=0,\ \frac{dw}{dx}(x=0)=0 \\ \text{Free End B.C.'s: none} \end{cases}$

Solve to get expression for w:
⇒ use Laplace; or use trial solution $w = A + Bx + Cx^2 + Dx^3$, then apply B.C.'s

$$w = \frac{FL}{2EI}x^2\left(1 - \frac{x}{3L}\right)$$ [Deflection @ x due to a point load F applied at $x=L$]

Cantilever Beam w/ a Concentrated Load



Maximum deflection @ $x=L$:

$$w_{max} = \left(\frac{L^3}{3EI}\right)F \rightarrow F = \left(\frac{3EI}{L^3}\right)w(x=L) = k_c w(x=L)$$

Note that in general, stiffness is a function of location x.

where $k_c = \frac{3EI}{L^3} \triangleq$ stiffness @ location $x=L$

$$\left[I = \frac{1}{12}Wh^3\right] \Rightarrow \boxed{k_c = \frac{1}{4}EW\frac{h^3}{L^3}}$$

Ex. $L = 100\mu m$, $W = 2\mu m$, $h = 2\mu m$
polysilicon → $E = 150\,GPa$

$$k_c = \frac{1}{4}(150G)(2\mu)\left(\frac{2\mu}{100\mu}\right)^3 = \underline{0.6\ N/m}$$

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Maximum Stress in a Bent Cantilever

From before, the radius of curvature is given by

$$\frac{1}{R} = \frac{d^2w}{dx^2} = \frac{F}{EI}(L-x) \Rightarrow \text{maximized where } R \to 0$$

↳ occurs at the support, where $x=0$:

$$[x=0] \Rightarrow \frac{1}{R} = \frac{d^2w}{dx^2} = \frac{FL}{EI}$$

Strain is maximized:

① At top surface → tensile
② At bottom surface → compressive

$$\epsilon_{max} = -\frac{z}{R} = \frac{h}{2}\frac{1}{R} = \frac{h}{2}\frac{FL}{EI}$$

$$\left[I = \frac{1}{12}Wh^3\right] \Rightarrow \epsilon_{max} = \frac{\cancel{h}}{2}\frac{FL}{E}\frac{12}{Wh^{\cancel{3}2}} = \frac{6L}{EWh^2}F$$

$$\therefore \boxed{\sigma_{max} = \epsilon_{max}E = \frac{6L}{Wh^2}F}$$

(Maximum Stress in a Bent Cantilever)

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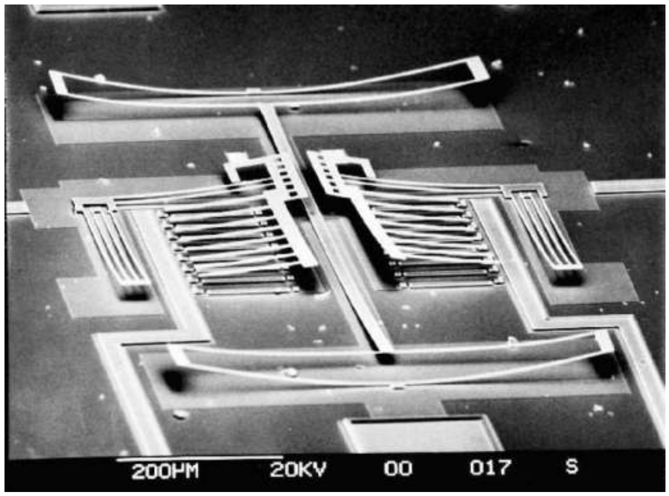
Stress Gradients in Cantilevers

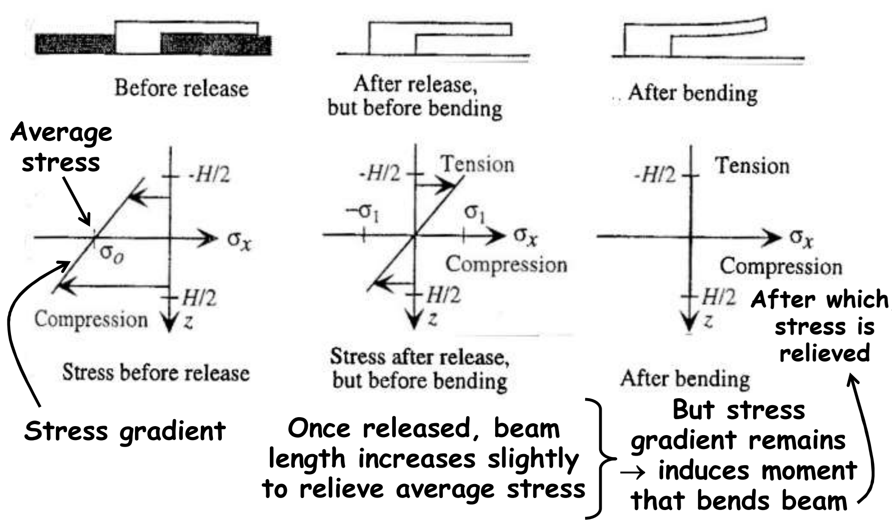
Vertical Stress Gradients

- Variation of residual stress in the direction of film growth
- Can warp released structures in z-direction



Stress Gradients in Cantilevers

- Below: surface micromachined cantilever deposited at a high temperature then cooled → assume compressive stress



Stress Gradients in Cantilevers (cont)

Find the radius of curvature.

Prior to release, axial stress is: $\sigma = \sigma_0 - \frac{\sigma_1}{(H/2)} z$

The internal moment:

$$M_x = \int_{-H/2}^{H/2} \left[(W \cdot dz)\sigma\right] \cdot z = W \int_{-H/2}^{H/2} \left(z\sigma_0 - \frac{\sigma_1 z^2}{(H/2)}\right) dz = W\left(\frac{1}{2}\sigma_0 z^2 - \frac{2\sigma_1 z^3}{3H}\right)\Bigg|_{-H/2}^{H/2}$$

$$= W\left(\cancel{\frac{1}{2}\sigma_0 \frac{H^2}{4}} - \frac{2}{3}\sigma_1 \frac{H^2}{8} - \cancel{\frac{1}{2}\sigma_0 \frac{H^2}{4}} - \frac{2\sigma_1 H^2}{3(8)}\right) = \underline{-\frac{1}{6}\sigma_1 W H^2 = M_x}$$

Thus, the radius curvature is:

$$\frac{1}{R} = -\frac{M_x}{E'I} \rightarrow R = -\frac{E'I}{M_x} = -\frac{E'\left(\frac{1}{12}WH^3\right)}{-\frac{1}{6}\sigma_1 W H^2} = \frac{1}{2}\frac{E'H}{\sigma_1}$$

Biaxial Stress Gradient

$\left[I = \frac{1}{12}Wh^3\right]$

$$\Rightarrow \boxed{R = \frac{1}{2}\frac{E}{(1-\nu)}\frac{H}{\sigma_1}}$$

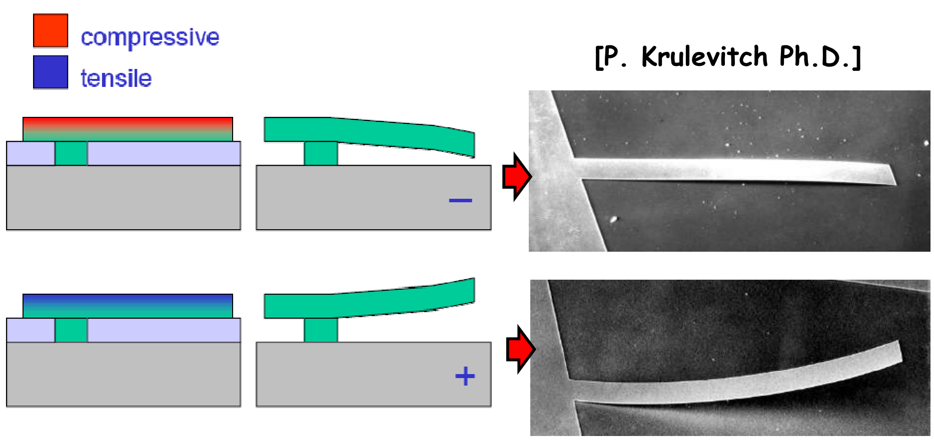
[Radius of Curvature for a Cantilever w/ Stress Gradient]

$$\boxed{\sigma_1 = \frac{1}{2}\frac{E}{(1-\nu)}\frac{H}{R}}$$

⇒ R can be used to determine stress gradient

Measurement of Stress Gradient

- Use cantilever beams
  - Strain gradient (Γ = slope of strain-thickness curve) causes beams to deflect up or down
  - Assuming linear strain gradient Γ, $z = \Gamma L^2/2$



[P. Krulevitch Ph.D.]

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Folded-Flexure Suspensions

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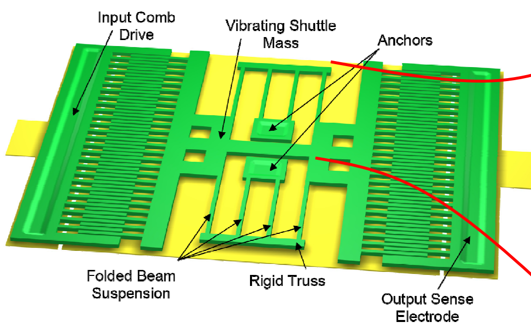
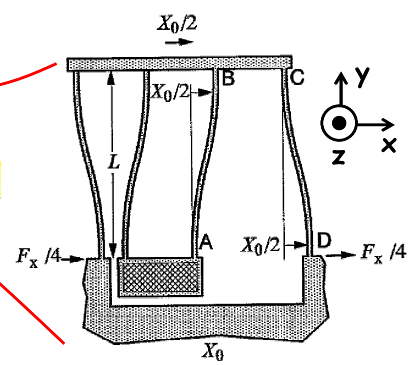
Folded-Beam Suspension

- Use of folded-beam suspension brings many benefits
  - Stress relief: folding truss is free to move in y-direction, so beams can expand and contract more readily to relieve stress
  - High y-axis to x-axis stiffness ratio



Comb-Driven Folded Beam Actuator

Folding Truss



Beam End Conditions

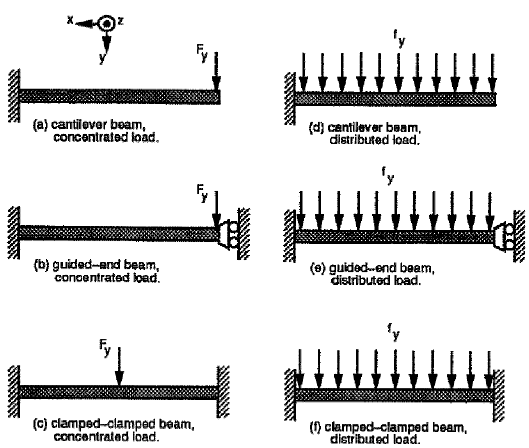
TABLE 4.1
Types of commonly used support conditions for beams and frames

Type of support	Displacement boundary conditions	Force boundary conditions
FREE	None	All, as specified
PINNED	$u = 0$ $w = 0$	Moment is specified
ROLLER (vertical)	$u = 0$	Transverse force and moment are specified
ROLLER (horizontal)	$w = 0$	Horizontal force and bending moment are specified
FIXED or CLAMPED	$u = 0$ $w = 0$ $dw/dx = 0$	None specified

[From Reddy, Finite Element Method]

Common Loading & Boundary Conditions

- Displacement equations derived for various beams with concentrated load F or distributed load f
- Gary Fedder Ph.D. Thesis, EECS, UC Berkeley, 1994



cantilever	guided-end	fixed-fixed
$x = \frac{F_x L}{Ehw}$	$x = \frac{F_x L}{Ehw}$	$x = \frac{F_x L}{4Ehw}$
$y = 4\frac{F_y}{Eh}\frac{L^3}{w^3}$	$y = \frac{F_y}{Eh}\frac{L^3}{w^3}$	$y = \frac{1}{16}\frac{F_y}{Eh}\frac{L^3}{w^3}$
$z = 4\frac{F_z}{Ew}\frac{L^3}{h^3}$	$z = \frac{F_z}{Ew}\frac{L^3}{h^3}$	$z = \frac{1}{16}\frac{F_z}{Ew}\frac{L^3}{h^3}$

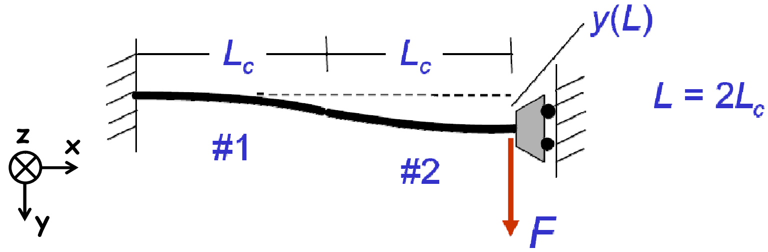
(a) Concentrated load.

cantilever	guided-end	fixed-fixed
$x = \frac{f_x L}{E}$	$x = \frac{f_x L}{E}$	$x = \frac{f_x L}{4E}$
$y = \frac{3}{2}\frac{f_y}{Eh}\frac{L^4}{w^3}$	$y = \frac{1}{2}\frac{f_y}{Eh}\frac{L^4}{w^3}$	$y = \frac{1}{32}\frac{f_y}{Eh}\frac{L^4}{w^3}$
$z = \frac{3}{2}\frac{f_z}{Ew}\frac{L^4}{h^3}$	$z = \frac{1}{2}\frac{f_z}{Ew}\frac{L^4}{h^3}$	$z = \frac{1}{32}\frac{f_z}{Ew}\frac{L^4}{h^3}$

(b) Distributed load.

Series Combinations of Springs

- For springs in series w/ one load
  - Deflections add
  - Spring constants combine like "resistors in parallel"



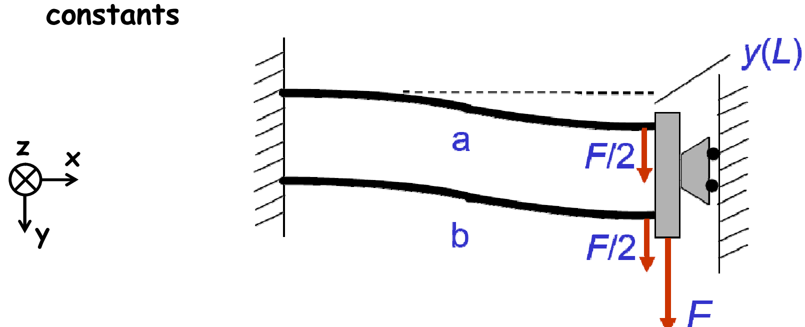
$$Y(L) = F/k = 2\,y(L_c) = 2\,(F/k_c) = F(1/k_c + 1/k_c)$$

Compliances effectively add:

$$\boxed{1/k = 1/k_c + 1/k_c} \rightarrow \boxed{k = k_c || k_c}$$

Parallel Combinations of Springs

- For springs in parallel w/ one load
  - Load is shared between the two springs
  - Spring constant is the sum of the individual spring constants

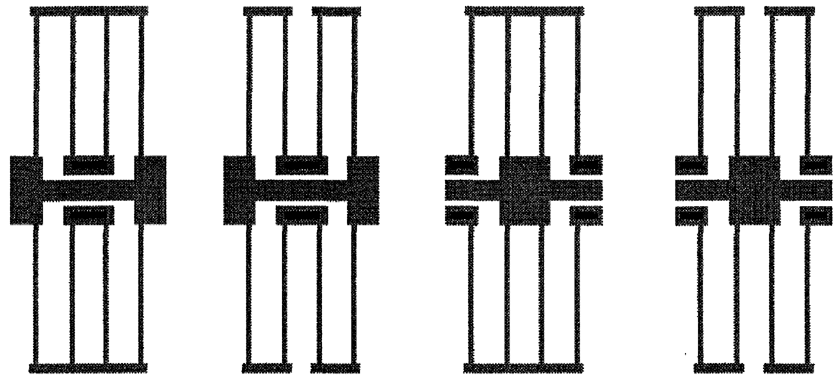


$$Y(L) = F/k = F_a/k_a = F_b/k_b = (F/2)\,(1/k_a)$$

$$\boxed{k = 2\,k_a}$$

Folded-Flexure Suspension Variants

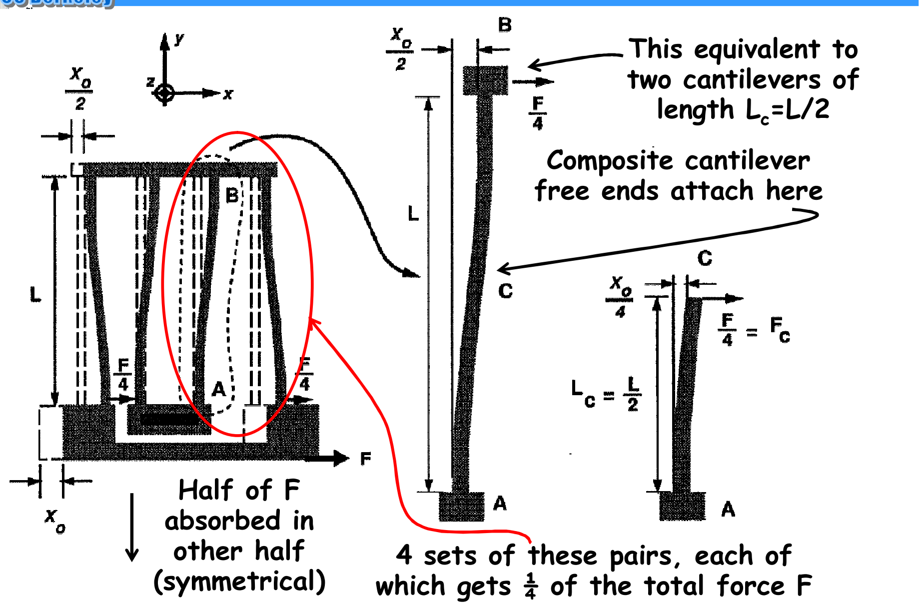
- Below: just a subset of the different versions
- All can be analyzed in a similar fashion



(a) Inner fold, continuous truss (b) Inner fold, discontinuous truss (c) Outer fold, continuous truss (d) Outer fold, discontinuous truss

[From Michael Judy, Ph.D. Thesis, EECS, UC Berkeley, 1994]

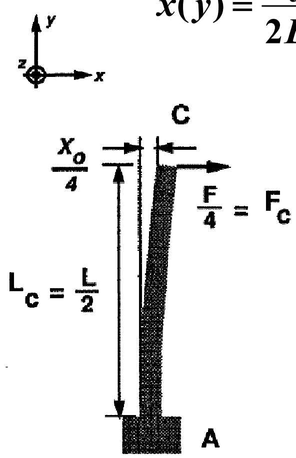
Deflection of Folded Flexures



Constituent Cantilever Spring Constant

- From our previous analysis:

$$x(y) = \frac{F_c L_c}{2EI_z} y^2 \left(1 - \frac{y}{3L_c}\right) = \frac{F_c y^2}{6EI_z}(3L_c - y)$$



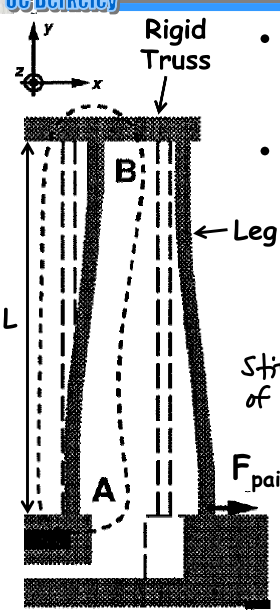
- From which the spring constant is:

$$k_c = \frac{F_c}{x(L_c)} = \frac{3EI_z}{L_c^3}$$

- Inserting L_c = L/2

$$\boxed{k_c = \frac{3EI_z}{(L/2)^3} = \frac{24EI_z}{L^3}}$$

Overall Spring Constant



- Four pairs of clamped-guided beams
  - In each pair, beams bend in series
  - (Assume trusses are inflexible)
- Force is shared by each pair → F_{pair} = F/4

total force on shuttle

Leg → Displacement of two legs add
↳ thus, springs are in series:

$$x = \frac{F_{pair}}{k_{pair}} = \frac{F_{pair}}{(k_{leg} \| k_{leg})} = \left(\frac{F}{4}\right)\left(\frac{1}{k_{leg}} + \frac{1}{k_{leg}}\right)$$

Stiffness of Pair

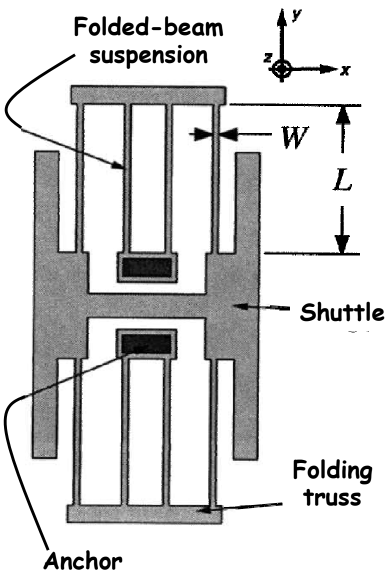
From before: $k_{leg} = k_c \| k_c = \frac{k_c}{2}$

Thus:

$$x = \left(\frac{F}{4}\right)\left(\frac{2}{k_c} + \frac{2}{k_c}\right) = \frac{F}{k_c} = \frac{F}{k_{tot}}$$

$$\Rightarrow \boxed{k_{tot} = k_c = \frac{24EI_z}{L^3}}$$

Folded-Beam Stiffness Ratios



- In the x-direction:

$$k_x = \frac{24EI_z}{L^3}$$

- In the z-direction:
  - Same flexure and boundary conditions

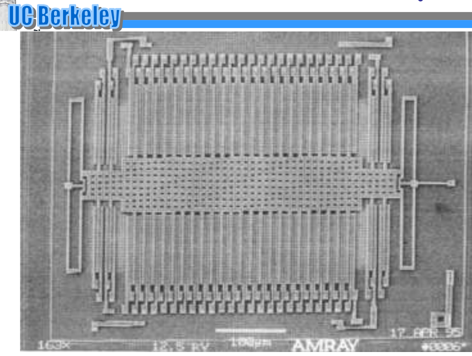
$$k_z = \frac{24EI_x}{L^3}$$

- In the y-direction:

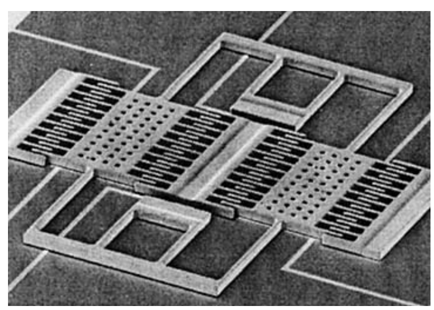
[See Senturia, §9.2] $$k_y = \frac{8EWh}{L}$$

- Thus: $$\boxed{\frac{k_y}{k_x} = 4\left(\frac{L}{W}\right)^2}$$ Much stiffer in y-direction!

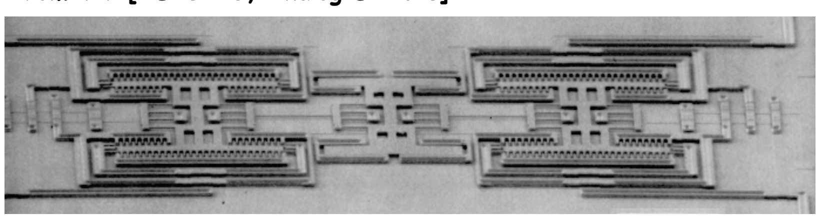
Folded-Beam Suspensions Permeate MEMS



Accelerometer [ADXL-05, Analog Devices]



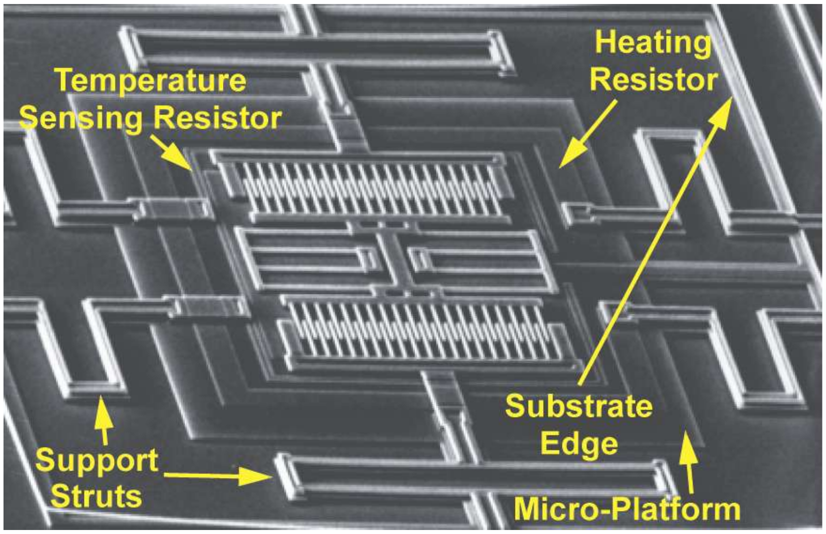
Gyroscope [Draper Labs.]



Micromechanical Filter [K. Wang, Univ. of Michigan]

Folded-Beam Suspensions Permeate MEMS

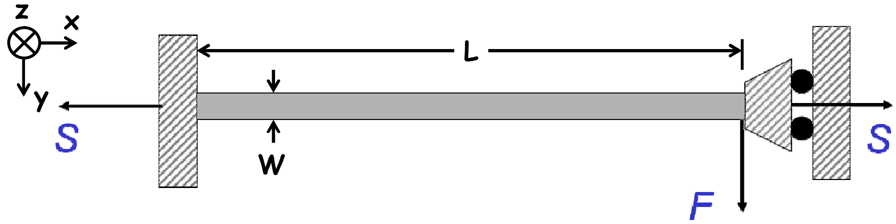
- Below: Micro-Oven Controlled Folded-Beam Resonator



Stressed Folded-Flexures

Clamped-Guided Beam Under Axial Load

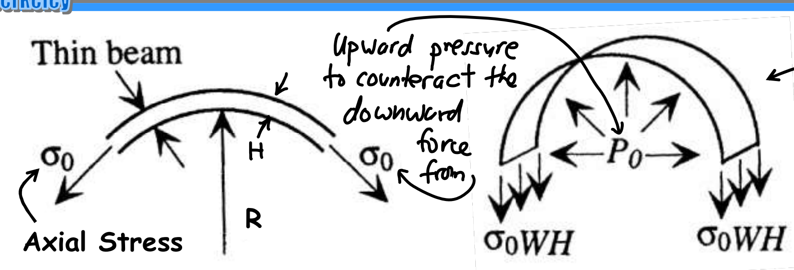
- Important case for MEMS suspensions, since the thin films comprising them are often under residual stress
- Consider small deflection case: y(x) « L



Governing differential equation: (Euler Beam Equation)

$$EI_z \frac{d^4 y}{dx^4} - \underset{\text{Axial Load}}{S} \frac{d^2 y}{dx^2} = \underbrace{F\delta(x-L)}_{\text{Unit impulse @ x=L}}$$

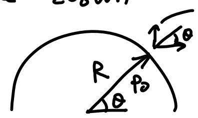
The Euler Beam Equation



- Axial stresses produce no net horizontal force; but as soon as the beam is bent, there is a net downward force
  - ↳ For equilibrium, must postulate some kind of upward load on the beam to counteract the axial stress-derived force
  - ↳ For ease of analysis, assume the beam is bent to angle π

Downward Vertical Force = $2\sigma_0 WH$

Upward Force due to P_0:



$P_y(\theta) = P_0 \sin\theta$

$$F_u = \int_0^\pi (P_0 \sin\theta) W (R d\theta) = -P_0 W R \cos\theta \Big|_0^\pi = 2RWP_0$$

The Euler Beam Equation

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[Equilibrium] $\Rightarrow 2RWP_0 = 2\sigma_0 WH \longrightarrow P_0 = \frac{\sigma_0 H}{R}$

$\left[q_0 = \frac{\text{beam load}}{\text{unit length}} = P_0 W,\ \frac{1}{R} = \frac{d^2 w}{dx^2}\right] \Rightarrow q_0 = \sigma_0 WH \frac{d^2 w}{dx^2}$

beam displacement

Using the differential beam bending equation:

$$\frac{d^2 w}{dx^2} = -\frac{M}{EI} \longrightarrow \frac{d^4 w}{dx^4} = \frac{q}{EI} \leftarrow \frac{\text{load}}{\text{unit length}}$$

external load

$$EI \frac{d^4 w}{dx^4} = q + q_0$$

equiv. load accounting for the axial stress contribution to the bending stiffness

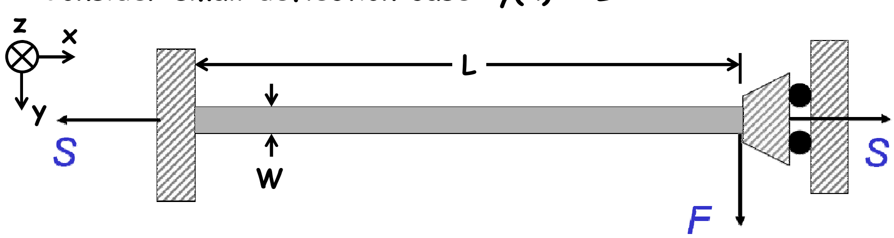
Note: Use of the full bend angle of π to establish conditions for load balance; but this returns us to case of small displacements and small angles

$$\left[q_0 = \sigma_0 WH \frac{d^2 w}{dx^2}\right] \Rightarrow EI \frac{d^4 w}{dx^4} - \underbrace{(\sigma_0 WH)}_{\text{tension in the beam} = S \leftarrow \text{a force}} \frac{d^2 w}{dx^2} = q \quad \text{[Euler Beam Equation]}$$

Clamped-Guided Beam Under Axial Load

UC Berkeley

- Important case for MEMS suspensions, since the thin films comprising them are often under residual stress
- Consider small deflection case: y(x) « L



Governing differential equation: (Euler Beam Equation)

$$EI_z \frac{d^4 y}{dx^4} - S \frac{d^2 y}{dx^2} = F \underbrace{\delta(x-L)}_{\text{Unit impulse @ } x=L}$$

Axial Load

Solving the ODE

- Can solve the ODE using standard methods
  - Senturia, pp. 232-235: solves ODE for case of point load on a clamped-clamped beam (which defines B.C.'s)
  - For solution to the clamped-guided case: see S. Timoshenko, Strength of Materials II: Advanced Theory and Problems, McGraw-Hill, New York, 3rd Ed., 1955
- Result from Timoshenko:

S > 0 (tension) $$k^{-1} = \frac{pL - 2\tanh(pL/2)}{p\,S|} = \frac{y(x=L)}{F}$$

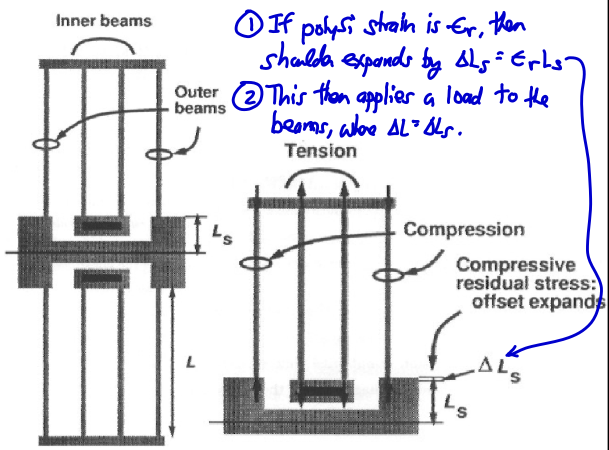
S < 0 (compression)

$$k^{-1} = \frac{-pL + 2\tan(pL/2)}{p|S|} = \frac{y(x=L)}{F}$$

where $p = \sqrt{\frac{|S|}{EI_z}}$

Design Implications

- Straight flexures
  - Large tensile S means flexure behaves like a tensioned wire (for which $k^{-1} = L/S$)
  - Large compressive S can lead to buckling ($k^{-1} \to \infty$)
- Folded flexures
  - Residual stress only partially released
  - Length from truss to shuttle's centerline differs by L_s for inner and outer legs



① If polySi strain is ϵ_r, then shuttle expands by $\Delta L_s = \epsilon_r L_s$

② This then applies a load to the beams, where $\Delta L = \Delta L_s$.

③ Beam strain:
$\epsilon_b = \frac{\Delta L}{2L} = \frac{\Delta L_s}{2L} = \epsilon_r \frac{L_s}{2L}$

over

Effect on Spring Constant

- Residual compression on outer legs with same magnitude of tension on inner legs:

strain in the polySi

Beam Strain: $\varepsilon_b = \pm\varepsilon_r\left(\frac{L_s}{2L}\right)$; Stress Force: $S = \pm E\varepsilon_r\left(\frac{L_s}{2L}\right)Wh$

strain in the beams

Expansion of the Shoulder $= \Delta L_s = \epsilon_r L_s$ ← This expansion applies a load on the beams of the flexure

- Spring constant becomes:

$\epsilon_b = \frac{\Delta L}{2L} = \frac{\epsilon_r L_s}{2L}$

$[\Delta L = \Delta L_s]$

$\epsilon_b = \epsilon_r \frac{L_s}{2L}$

$$k = 4(k_{com}^{\;-1} + k_{ten}^{\;-1})^{-1}$$

$$k = 4\left[\frac{-pL + 2\tan(pL/2)}{p|S|} + \frac{pL - 2\tanh(pL/2)}{p|S|}\right]^{-1}$$

- <u>Remedies</u>:
  - Reduce the shoulder width L_s to minimize stress in legs
  - Compliance in the truss lowers the axial compression and tension and reduces its effect on the spring constant