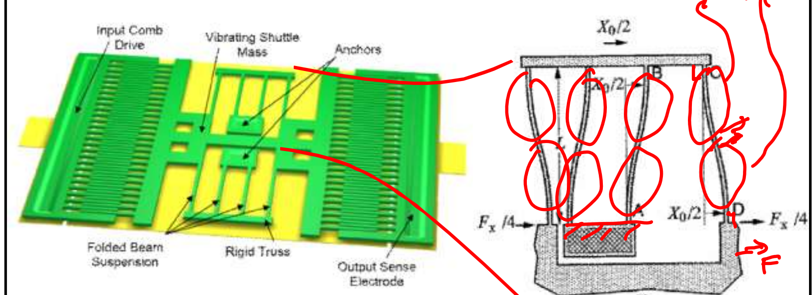
Lecture 12: Beam Bending I

- Announcements:
- HW#2 online and due Tuesday, 2/28, at 10 a.m.
- Module 8 on "Microstructural Elements" online
- Graded HW#1 handed back
- ------------------------------------
- Reading: Senturia, Chpt. 8
- Lecture Topics:
  - Stress, strain, etc., for isotropic materials
  - Thin films: thermal stress, residual stress, and stress gradients
  - Internal dissipation
  - MEMS material properties and performance metrics
- ---------------------------------
- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients
- ------------------------------------
- Last Time:
- Nearly finished with Material in Module 7
- Then, start a new topic: Bending of Beams

- Springs and suspensions very common in MEMS
- Coils are popular in the macro-world; but not easy to make in the micro-world
- Beams: simpler to fabricate and analyze; become "stronger" on the micro-scale → use beams for MEMS

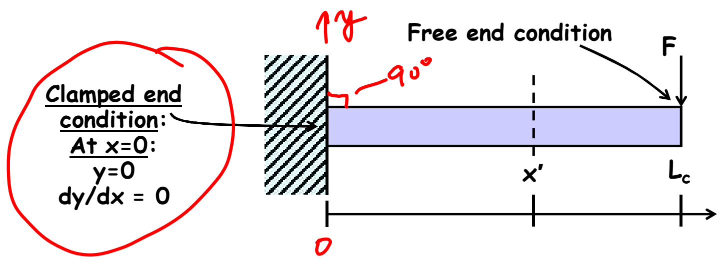
$x = \frac{F}{k}$ $(F = kx)$

cantilever

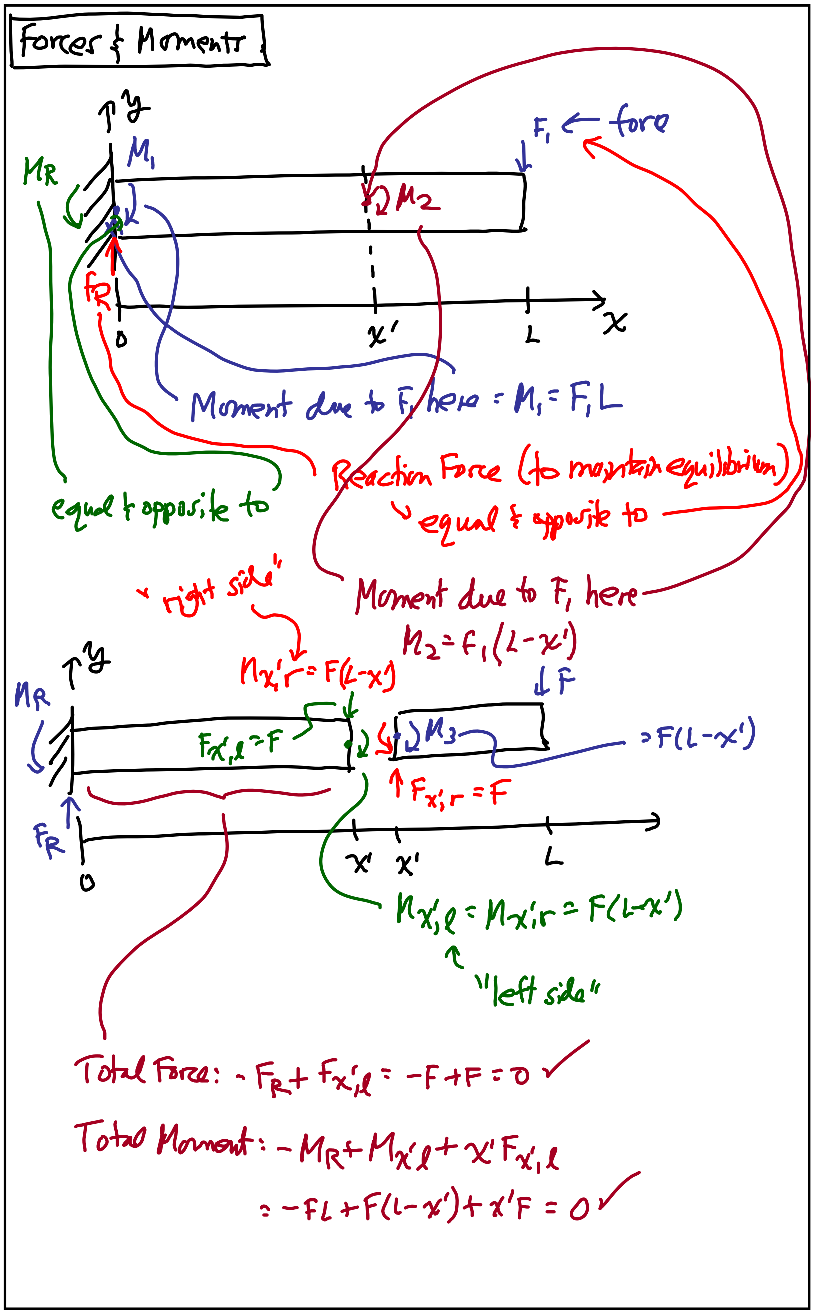


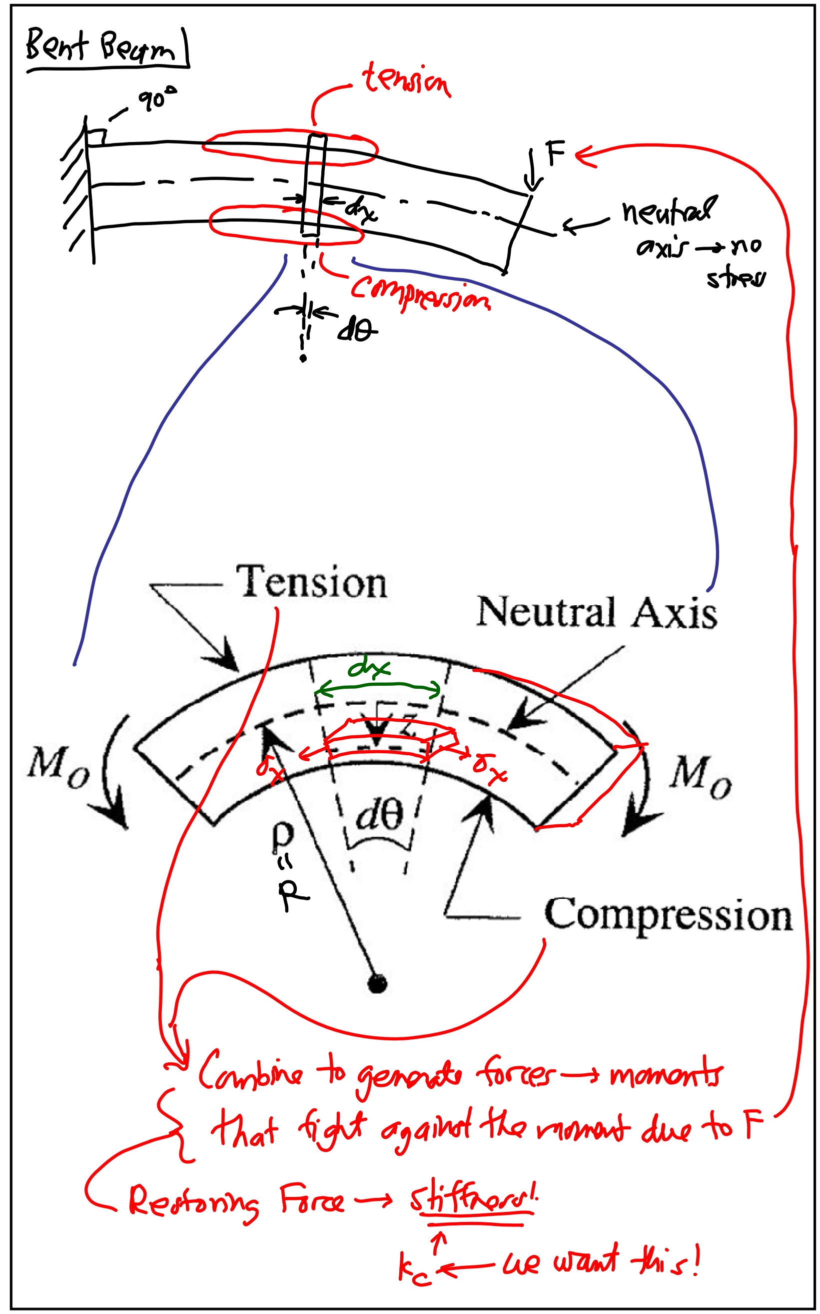
Comb-Driven Folded Beam Actuator

Problem: Bending a Cantilever Beam



- Objective: Find relation between tip deflection $y(x=L_c)$ and applied load F
- Assumptions:
  1. Tip deflection is small compared with beam length
  2. Plane sections (normal to beam's axis) remain plane and normal during bending, i.e., "pure bending"
  3. Shear stresses are negligible


Forces & Moments
y
M_R
M_1
M_2
F_1 ← force
F_R
0
x'
L
x
Moment due to F_1 here = $M_1 = F_1 L$
Reaction Force (to maintain equilibrium)
equal & opposite to
equal & opposite to
Moment due to F_1 here
$M_2 = F_1(L - x')$
"right side"
$M_{x',r} = F(L-x')$
$F_{x',l} = F$
M_3
= $F(L-x')$
F
$F_{x',r} = F$
$M_{x',l} = M_{x',r} = F(L-x')$
"left side"
Total Force: $-F_R + F_{x',l} = -F + F = 0$ ✓
Total Moment: $-M_R + M_{x',l} + x' F_{x',l}$
$= -FL + F(L-x') + x'F = 0$ ✓


Bent Beam
90°
tension
F
dx
neutral axis → no stress
compression
dθ
Tension
Neutral Axis
M_o
M_o
dx
δ_x
δ_x
dθ
ρ = R
Compression
Combine to generate forces → moments that fight against the moment due to F
Restoring Force → stiffness!
k_c ← we want this!

Beam Segment in Pure Bending

⇒ consider the segment bounded by the dashed lines defining $d\theta$

At $z=0$: neutral axis → segment length $= dx = R\,d\theta$ (1)

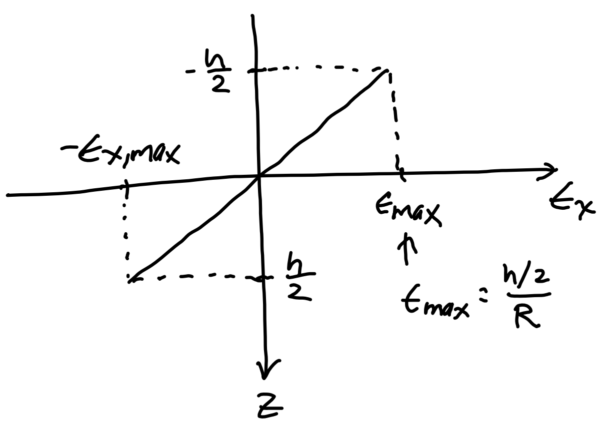
At any z: segment length $= dL = (R-z)\,d\theta$ (2)

Combine (1) & (2): $dL = dx - z\,d\theta = dx - \frac{z}{R}dx$

Thus, the axial strain @ z:

$$\epsilon_x = \frac{dL - dx}{dx} = -\frac{z}{R} \rightarrow \boxed{\epsilon_x = -\frac{z}{R}}$$

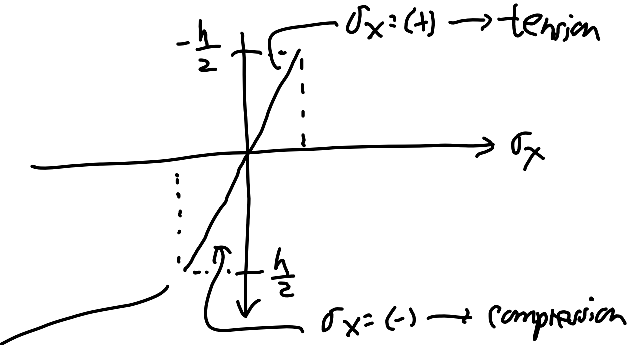
Thus, the strain varies linearly along the beam thickness:



$$\epsilon_{max} = \frac{h/2}{R}$$

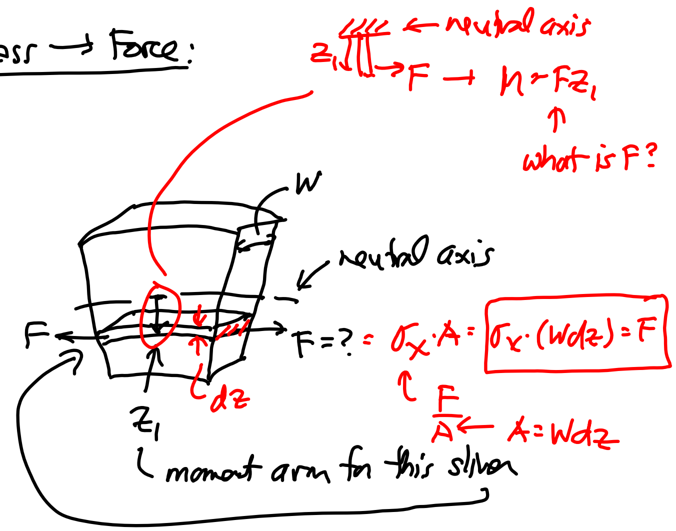
Of course, there is a corresponding axial stress:

$$\sigma_x = \epsilon_x E = \boxed{-\frac{zE}{R} = \sigma_x}$$



This gradient of stress then generates a bending moment! → in response to the original applied moment (from F)

Stress → Force:



⇒ integrate forces due to stresses through the beam thickness to get the total moment:

$$M = \int_{-\frac{h}{2}}^{\frac{h}{2}} \underbrace{[(W dz)\sigma_x]}_{\text{force}} z$$

$$= -\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{EWz^2}{R} dz \Rightarrow \boxed{M = -\left(\frac{1}{12}Wh^3\right)\frac{E}{R}}$$

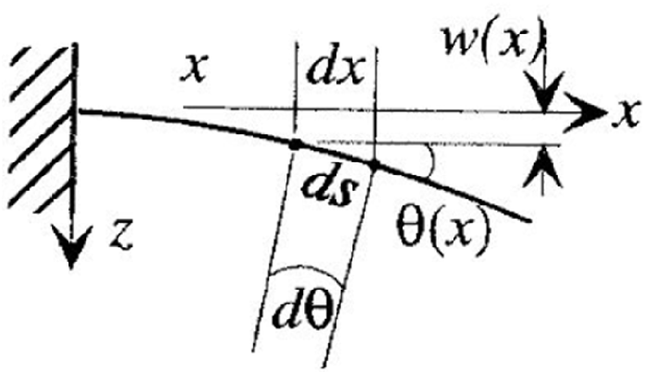
$$\left[\sigma_x = -\frac{zE}{R}\right]$$

$\frac{1}{12}Wh^3 = I \triangleq$ Moment of Inertia

$$\boxed{\frac{1}{R} = -\frac{M}{EI}}$$

Note: (+) radius of curvature → (−) internal bending moment

Differential Equation for Beam Bending



Write out some geometric relationships:

⇒ then use small angle approx:

$$\cos\theta = \frac{dx}{ds} \rightarrow ds = \frac{dx}{\cos\theta} \longrightarrow ds \cong dx$$

$$\tan\theta = \frac{dw}{dx} = \text{slope of the beam @ this point} \longrightarrow \theta \cong \frac{dw}{dx} \quad (1)$$

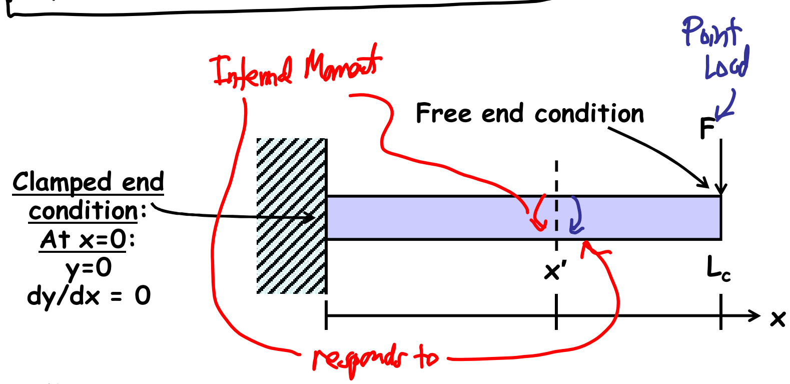
$$ds = R d\theta \rightarrow \frac{1}{R} = \frac{d\theta}{ds} \longrightarrow \frac{1}{R} = \frac{d\theta}{dx} \quad (2)$$

Inserting (1) into (2):

$$\frac{1}{R} = \boxed{\frac{d^2w}{dx^2} = -\frac{M}{EI}}$$

← Diff. Eq. for Small Angle Beam Bending

Cantilever Beam w/ Concentrated Load



Internal Moment @ position x: $M = -F(L-x)$

Thus:

$$\frac{d^2w}{dx^2} = \frac{F}{EI}(L-x)$$