Lecture 15: Beam Combos II

- **Announcements:**
- HW#3 online, due Thursday, next week, 10 a.m.
  - shorter time span than before
- Midterm Exam less than 3 weeks away, Tuesday, March 21, 3:30-5 p.m., 521 Cory (right here)
- ------------------------------------
- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients for folded-beam devices
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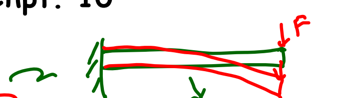
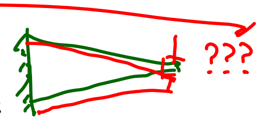
- Reading: Senturia, Chpt. 10
- Lecture Topics:
  - Energy Methods
  - Virtual Work
  - Energy Formulations
  - Tapered Beam Example
  - Estimating Resonance Frequency





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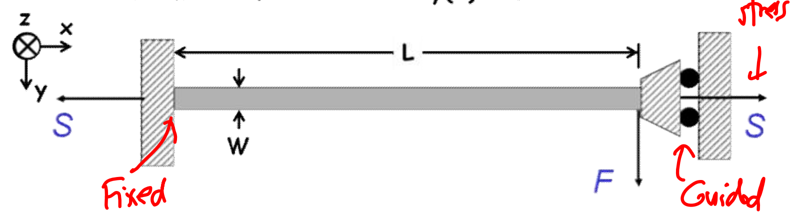
- **Last Time:**
- Finished mechanical filter spring combo example
- Continue with Module 8 on Microstructural Elements
- Go through slides 31-33

Tensioned Spring (Non-Ideality)

- Important case for MEMS suspensions, since the thin films comprising them are often under residual stress
- Consider small deflection case: $y(x) \ll L$



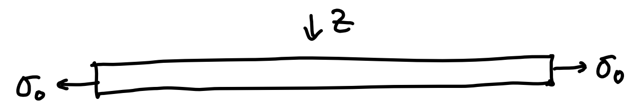
Governing differential equation: (Euler Beam Equation)

$$EI_z \frac{d^4 y}{dx^4} - \overset{\text{Force}}{S}\frac{d^2 y}{dx^2} = \underbrace{F\delta(x-L)}_{\text{Unit impulse @ x=L}}$$

Axial Load

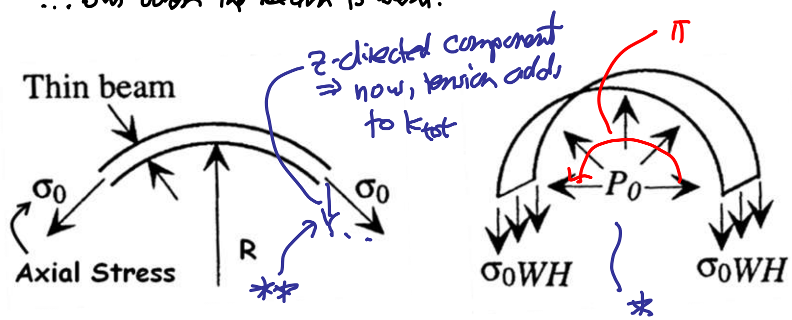
Heuristic Derivation for the Euler Beam Equation

Consider first a straight beam under axial stress:



⇒ tension has no effect on z-directed stiffness when the beam is straight

...but when the beam is bent:



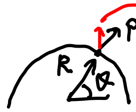
*← Upward pressure P_o to counteract the downward force from ** to keep everything in static equilibrium

For ease of analysis:

Assume the beam is bent to an angle π

↳ Downward force: $2\sigma_o W H$

Upward Force due to P_o:



$P_y(\theta) = P_o \sin\theta$

$$F_u = \int_0^{\pi} (P_o \sin\theta) W (R d\theta)$$
$$= -P_o W R \cos\theta \Big|_0^{\pi}$$
$$= 2RWP_o$$

[Equilibrium] ⇒ $2RWP_o = 2\sigma_o W H \longrightarrow P_o = \frac{\sigma_o H}{R}$

$\left[q_o = \frac{\text{beam load}}{\text{unit length}} = P_o W, \quad \frac{1}{R} = \frac{d^2 w}{dx^2} \right]$ ← beam displacement

$q_o = \sigma_o W H \frac{d^2 w}{dx^2}$ ← generalizes to smaller displacements & angles

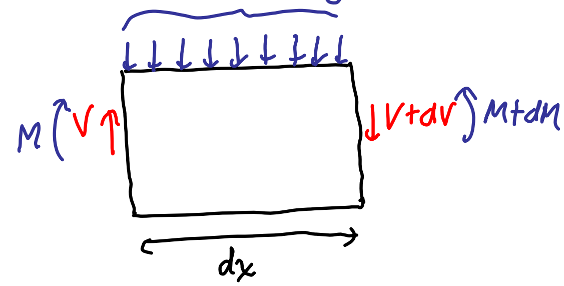
Using the differential beam bending equation:

$$\frac{d^2 w}{dx^2} = -\frac{M}{EI} \xrightarrow[???]{} \frac{d^4 w}{dx^4} = \frac{q}{EI} \leftarrow \frac{\text{load}}{\text{unit length}}$$

↳ $M = -EI \frac{d^2 w}{dx^2}$

Relationships Between Forces and Moments on a Fully Loaded Differential Beam Element

$q = \frac{\text{force}}{\text{unit length}}$



[Total Static Equilibrium] ⟹ total force = 0

$$F_T = \text{total force} = q\,dx + (V + dV) - V = 0$$

$$\therefore \boxed{\frac{dV}{dx} = -q} \qquad (1)$$

⇒ also, total moment w/r to the left hand edge = 0

$$M_T = (M + dM) - M - (V + \cancel{dV})dx - \cancel{\tfrac{1}{2} q\,dx^2} = 0$$

[neglect products of differentials] $\int_0^{dx} (q\,du)u = \frac{1}{2} q\,dx^2$

$$dM - V dx = 0 \longrightarrow \boxed{\frac{dM}{dx} = V} \qquad (2)$$

Using (1) & (2):

$$\left[\frac{d^2 M}{dx^2} = \frac{dV}{dx} = -q \right]$$

$$EI \frac{d^4 w}{x^4} = q + q_o$$

external load (q); q_o ↖ equiv. load f/ axial stress *

*↷ $\left[q_0 = \sigma_0 W H \frac{d^2 w}{dx^2}\right] \Rightarrow$

$$\boxed{EI\frac{d^4 w}{dx^4} - \underbrace{(\sigma_0 W H)}\frac{d^2 w}{dx^2} = q}$$

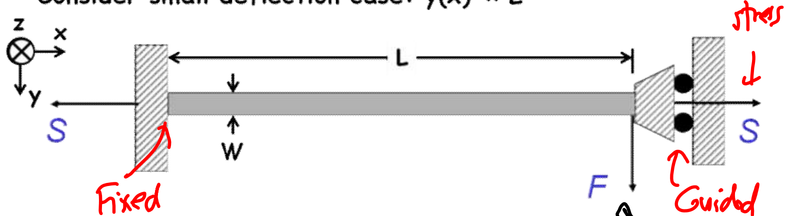
tension in the beam = S

a force

Euler Beam Equation

Clamped-Guided Beam Under Axial Load

- Important case for MEMS suspensions, since the thin films comprising them are often under residual stress
- Consider small deflection case: y(x) « L



Governing differential equation: (Euler Beam Equation)

Force

$$EI_z\frac{d^4 y}{dx^4} - S\frac{d^2 y}{dx^2} = \underbrace{F\delta(x-L)}_{\text{Unit impulse @ x=L}}$$

Axial Load

Need to solve this, then find the stiffness against this force (@ this location)

- Can solve the ODE using standard methods
  - Senturia, pp. 232-235: solves ODE for case of point load on a clamped-clamped beam (which defines B.C.'s)
  - For solution to the clamped-guided case: see S. Timoshenko, Strength of Materials II: Advanced Theory and Problems, McGraw-Hill, New York, 3rd Ed., 1955
- Result from Timoshenko:

S > 0 (tension) $\quad k^{-1} = \frac{pL - 2\tanh(pL/2)}{p|S|} = \frac{y(x=L)}{F}$ $\quad$ " k_{ten}^{-1}

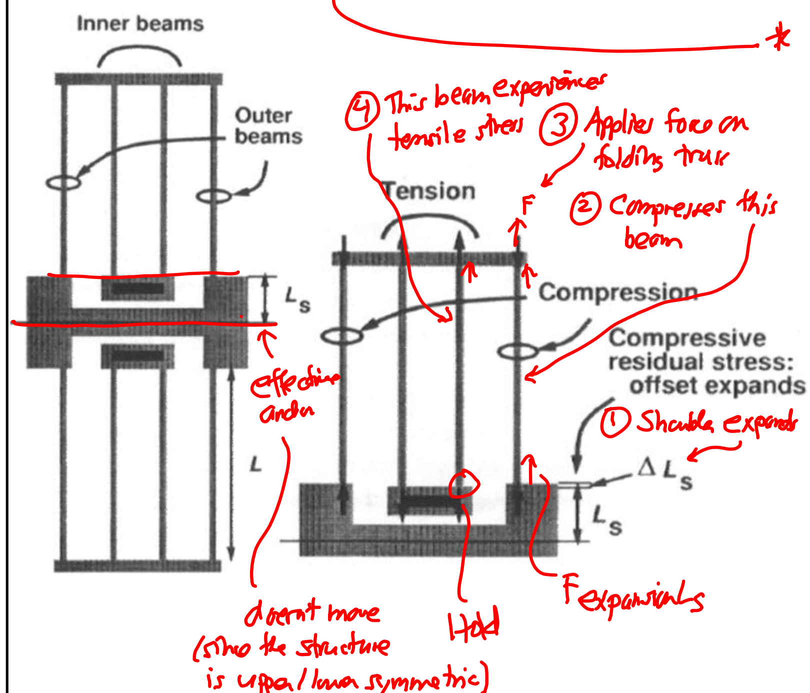
S < 0 (compression)

$$k_{comp}^{-1} = k^{-1} = \frac{-pL + 2\tan(pL/2)}{p|S|} = \frac{y(x=L)}{F}$$

where $p = \sqrt{\frac{|S|}{EI_z}}$

? → need to compute this

*

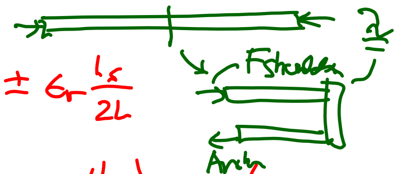


Get S: structural material

① If the polySi strain is ϵ_r, then the shuttle expands $\Delta L_s = \epsilon_r L_s$

② This then applies a load to the beams, $\Delta L = \Delta L_s$

③ Beam Stress:



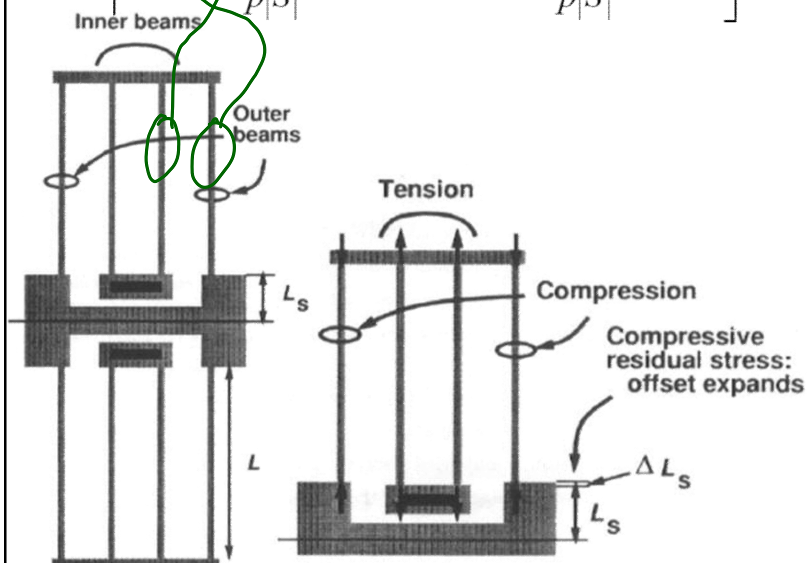
$$\epsilon_b = \frac{\Delta L}{2L} = \frac{\Delta L_s}{2L} = \pm \epsilon_r \frac{L_s}{2L}$$

Stress Force: $S = \pm E \epsilon_r \left(\frac{L_s}{2L}\right) W h$ (axial tension) *

④ Spring Constant:

$$k = 4\left(k_{com}^{-1} + k_{ten}^{-1}\right)^{-1} = k_{com} \| k_{ten}$$ series

$$k = 4\left[\frac{-pL + 2\tan(pL/2)}{p|S|} + \frac{pL - 2\tanh(pL/2)}{p|S|}\right]^{-1}$$



More General Geometries

- Euler-Bernoulli beam theory works well for simple geometries
- But how can we handle more complicated ones?
- Example: tapered cantilever beam
- Objective: Find an expression for displacement as a function of location x under a point load F applied at the tip of the free end of a cantilever with tapered width W(x)