Lecture 17: Resonance Frequency

- Announcements:
- Computer hard drive failed, so this is a backup computer … hopefully, it still records
- HW#4 online, due this Thursday, 10 a.m.
- Module 10 on "Resonance Frequency" online
- Midterm Exam next week, Thursday, March 23, 3:30-5 p.m., 521 Cory (right here)
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- Reading: Senturia, Chpt. 10: §10.5, Chpt. 19
- Lecture Topics:
  - Estimating Resonance Frequency
  - Lumped Mass-Spring Approximation
  - ADXL-50 Resonance Frequency
  - Distributed Mass & Stiffness
  - Folded-Beam Resonator
  - Resonance Frequency Via Differential Equations
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- Last Time:
- Started resonance frequency
- Continue with this

Estimating Resonance Frequency

$\mapsto x(t) = X_0 \cos\omega t$

Potential Energy:

$$\mathcal{W}(t) = \frac{1}{2}k\,x^2(t) = \frac{1}{2}k X_0^2 \cos^2\omega t$$

Kinetic Energy:

$$\mathcal{K}(t) = \frac{1}{2}m[\dot{x}(t)]^2 = \frac{1}{2}m X_0^2 \omega^2 \sin^2\omega t$$

$\dot{x} = \frac{dx}{dt}$ = velocity

Remarks.

① Energy must be conserved.

② Total Energy = Potential Energy + Kinetic Energy at all times & locations on the structure

$$\mathcal{W}_{max} = \frac{1}{2}kX_0^2 = \mathcal{K}_{max} = \frac{1}{2}m\omega^2 X_0^2$$

maximum potential energy

peak displacement ($\mathcal{K}=0$)

maximum kinetic energy

maximum velocity ($\mathcal{W}=0$)

*

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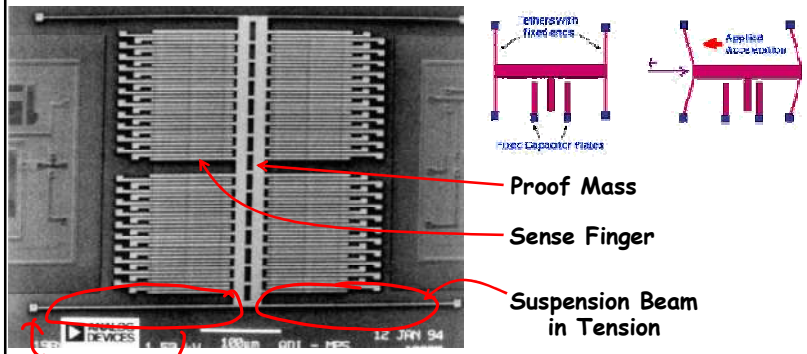
* $\boxed{\omega_0 = \sqrt{\frac{k}{m}}}$ ⇒ good for problems where mass & stiffness can be separated, i.e., are distinct

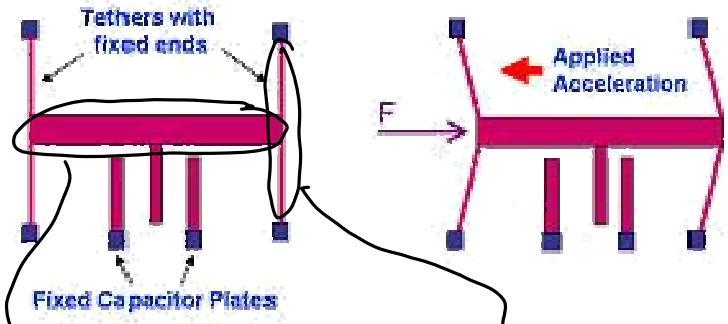
radian resonance frequency

Example. ADXL-50

- The proof mass of the ADXL-50 is many times larger than the effective mass of its suspension beams
  - ↳ Can ignore the mass of the suspension beams (which greatly simplifies the analysis)
- Suspension Beam: L = 260 μm, h = 2.3 μm, W = 2 μm



In fabrication, purposely introduce a tensile stress in the beams!
a rather large one!

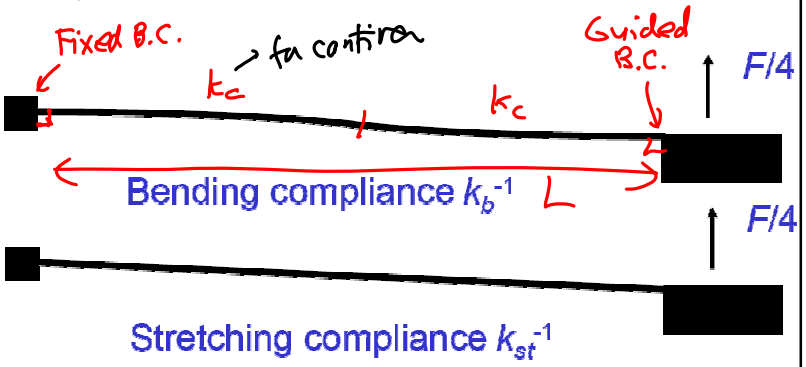


mass of structure >> mass of springs
∴ ignore the mass of springs

stiffness of springs << stiffness of the structure
∴ ignore the stiffness of the structure

For ADXL-50: 60% of the mass comes from the sense fingers → M = 162 ng

Suspension: Four tensioned beams



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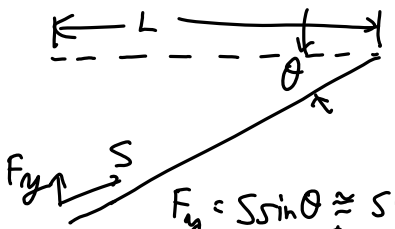
Bending Contribution

$$k_b = k_c \| k_c = \left(\frac{1}{k_c}+\frac{1}{k_c}\right)^{-1} = \frac{k_c}{2} = \frac{1}{2}\left(\frac{3E(Wh^3/12)}{(L/2)^2}\right)$$

$$\Rightarrow k_b = EW\left(\frac{h}{L}\right)^3 = 0.24\ N/m$$

Stretching Contribution



$$F_y = S\sin\theta \approx S\theta \approx S\left(\frac{y}{L}\right) = \underbrace{\left(\frac{S}{L}\right)}_{k_{st}} y$$

[assume small displacements]

k_{st} stretching stiffness

$$k_{st} = \frac{S}{L} = \frac{\sigma_r Wh}{L} = 0.88\ N/m$$

Get the total spring constant

bending stiffness
stretching stiffness } parallel → add!

$$k_{tot} = 4(k_b + k_{st}) = 4(0.24+0.88) = 4.5\ N/m$$

Now, get the resonance freq.:

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{4.48\ N/m}{162\times10^{-12}\ kg}} = 26.5\ kHz$$

ADXL-50 Data Sheet: $f_0 = 24\ kHz$ difference?

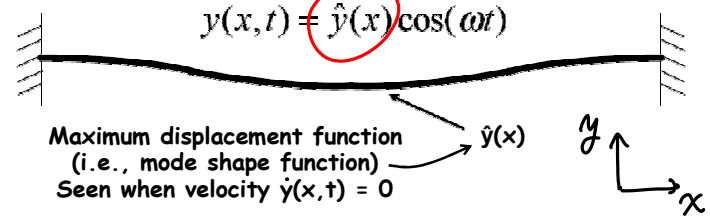
→ capacitive transducer → electrical stiffness

Find Resonance Frequency When Mass & Stiffness is Distributed

- Vibrating structure displacement function:

$$y(x,t) = \hat{y}(x)\cos(\omega t)$$

mode shape function

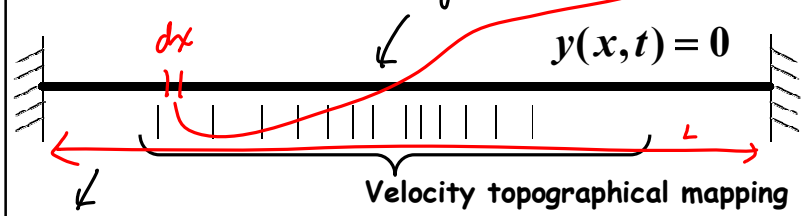


- Procedure for determining resonance frequency:
  - Use the static displacement of the structure as a trial function and find the strain energy W_{max} at the point of maximum displacement (e.g., when t=0, π/ω, …)
  - Determine the maximum kinetic energy when the beam is at zero displacement (e.g., when it experiences its maximum velocity)
  - Equate energies and solve for frequency

Get Maximum Kinetic Energy

velocity: $v(x,t) = \frac{\partial y(x,t)}{\partial t} = -\omega\hat{y}(x)\sin(\omega t)$

largest velocity



When $y(x,t)=0$, all the energy in the structure is kinetic ($W=0$) → $t = \frac{\pi}{2\omega}, \frac{3\pi}{2\omega}, \ldots$

$$v\left(x, \frac{(2m+1)\pi}{2\omega}\right) = -\omega\hat{y}(x)$$

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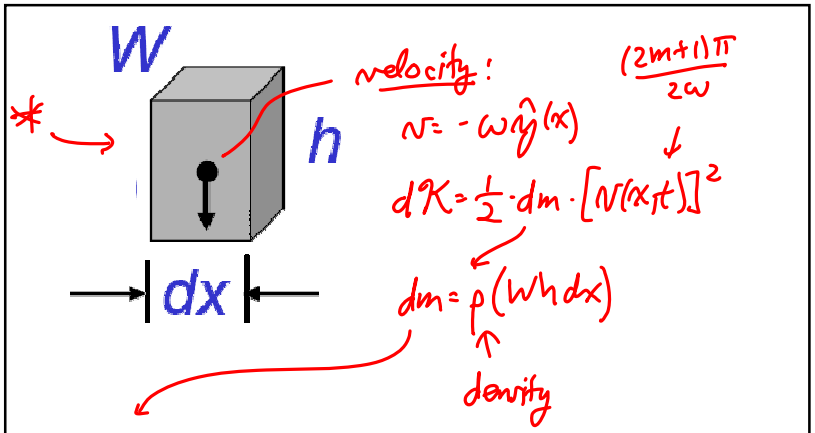


Maximum K.E., K_{max}:

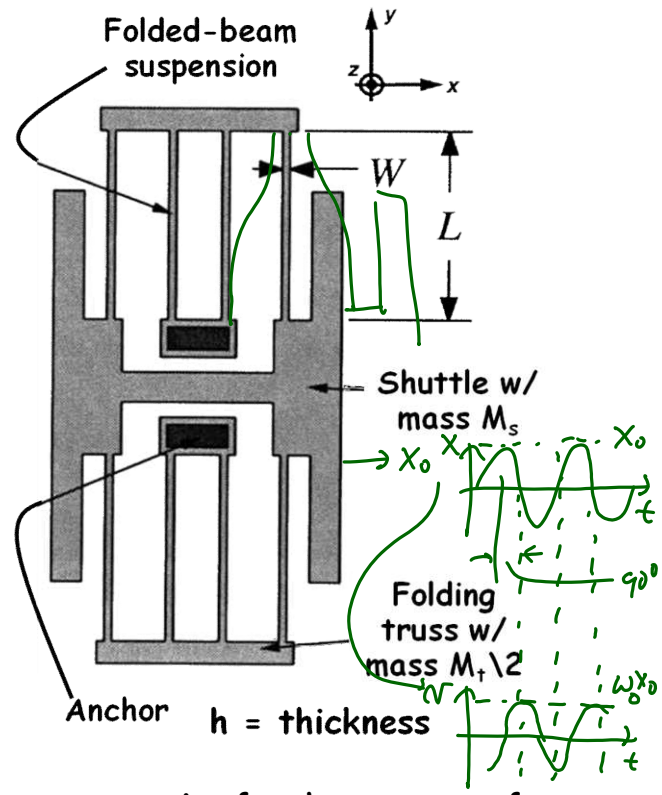
$$K_{max} = \int_0^L \frac{1}{2}\rho Whdx\, v^2(x,t) = \int_0^L \frac{1}{2}\rho Wh\omega^2[\hat{y}(x)]^2 dx$$

To get frequency: $K_{max} = W_{max}$

$$\therefore \omega = \sqrt{\frac{W_{max}}{\int_0^L \frac{1}{2}\rho Wh[\hat{y}(x)]^2 dx}} \quad [\text{radians/s}]$$

ω = radian resonance freq.
W_{max} = maximum potential energy
ρ = density of the structural material
W = beam width
h = " thickness
$\hat{y}(x)$ = resonance mode shape

Resonance Freq. of a Folded-Beam Resonator



- Derive an expression for the resonance frequency of the above structure

Use the Rayleigh-Ritz Method: (energy method)

$$K_{max} = W_{max} \leftarrow \frac{1}{2}kx^2$$

Find the kinetic energy → one piece @ a time

$$K_{max} = \underbrace{K_s}_{\text{shuttle}} + \underbrace{K_t}_{\text{truss}} + \underbrace{K_b}_{\text{beams}}$$

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$$\mathcal{K}_{max} = \frac{1}{2} v_s^2 M_s + \frac{1}{2} v_t^2 M_t + \frac{1}{2}\int v_b^2 \, dM_b$$

Velocity of Shuttle: $v_s = \omega_0 X_0$

res. freq. — maximum displacement of shuttle

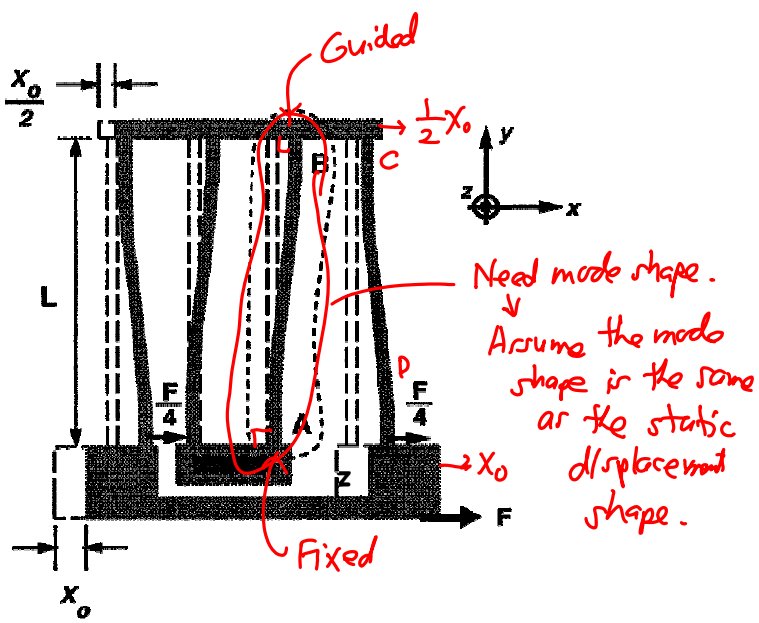
$$\therefore \mathcal{K}_s = \frac{1}{2} v_s^2 M_s = \frac{1}{2}\omega_0^2 X_0^2 M_s = \mathcal{K}_s$$

Velocity of Truss: $v_t = \frac{1}{2} v_s = \frac{1}{2}\omega_0 X_0$

$$\therefore \mathcal{K}_t = \frac{1}{2}\left(\frac{1}{2}\omega_0 X_0\right)^2 M_t = \frac{1}{8}\omega_0^2 X_0^2 M_t = \mathcal{K}_t$$

mass of both trusses

Velocity of the Beam Segments:



Segment [AB]:

$$\hat{x}(y) = \frac{F_x}{48EI_z}\left(3Ly^2 - 2y^3\right), \quad 0 \le y \le L \qquad (1)$$

At $y = L$: $x(L) = \frac{X_0}{2} = \frac{F_x L^3}{48EI_z}$ ← B.C.

Substitute into (1):

$$\hat{x}(y) = \frac{X_0}{2}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]$$

which yields for velocity

$$v_b(y)\Big|_{[AB]} = \frac{X_0}{2}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]\omega_0$$

Plugging into the expression for $\mathcal{K}_b$:

$$\mathcal{K}_{[AB]} = \frac{1}{2}\int_0^L \frac{X_0^2\omega_0^2}{4}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]^2 dM_{[AB]}$$

$$= \frac{X_0^2\omega_0^2 M_{[AB]}}{L}\int_0^L \left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]^2 dy$$

$M_{[AB]}$: static mass

mass per unit length

$$\mathcal{K}_{[AB]} = \frac{13}{280} X_0^2 \omega_0^2 M_{[AB]}$$

For segment [CD]:

$$v_b(y)\Big|_{[CD]} = X_0\left[1 - \frac{3}{2}\left(\frac{y}{L}\right)^2 + \left(\frac{y}{L}\right)^3\right]\omega_0$$

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Thus:

$$\mathcal{K}_{[CD]} = \frac{X_0^2 \omega_0^2 M_{[CD]}}{2L} \int_0^L \left[1 - \frac{3}{2}\left(\frac{y}{L}\right)^2 + \left(\frac{y}{L}\right)^3\right]^2 dy$$

$$\mathcal{K}_{[CD]} = \frac{83}{280} X_0^2 \omega_0^2 M_{[CD]}$$

static mass of beam [CD]

Let $M_b \triangleq$ total mass of all 8 beams

Thus:

$$\mathcal{K}_b = 4\mathcal{K}_{[AB]} + 4\mathcal{K}_{[CD]} = \frac{6}{35} X_0^2 \omega_0^2 M_b$$

and

$$\mathcal{K}_{max} = X_0^2 \omega_0^2 \left[\frac{1}{2} M_s + \frac{1}{8} M_t + \frac{6}{35} M_b\right]$$

both trusses

all beams

$\mathcal{W}_{max}$ → max potential energy → equal to the work done to achieve maximum deflection

$$\mathcal{W}_{max} = \frac{1}{2} k_x X_0^2$$

Then, using Rayleigh-Ritz:

$$\mathcal{K}_{max} = \mathcal{W}_{max}$$

$$X_0^2 \omega_0^2 \left[\frac{1}{2} M_s + \frac{1}{8} M_t + \frac{6}{35} M_b\right] = \frac{1}{2} \cancel{k_x} X_0^2$$

k_c

$$\boxed{\omega_0 = \left[\frac{k_c}{M_{eq}}\right]^{1/2} \quad \text{where } M_{eq} = M_s + \frac{1}{4} M_t + \frac{12}{35} M_b}$$

(Resonance Freq. of a Folded-Beam Suspended Shuttle)

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