Lecture 20: Electrical Stiffness

- Announcements:
- Module 12 on Capacitive Transducers online
- HW#5 online and due Thursday, April 13
- In mid-class (to make sure everyone is here)
  - Project introduction today
  - Midterm exams coming back today
  - Z-scores presented as well, at the end of class
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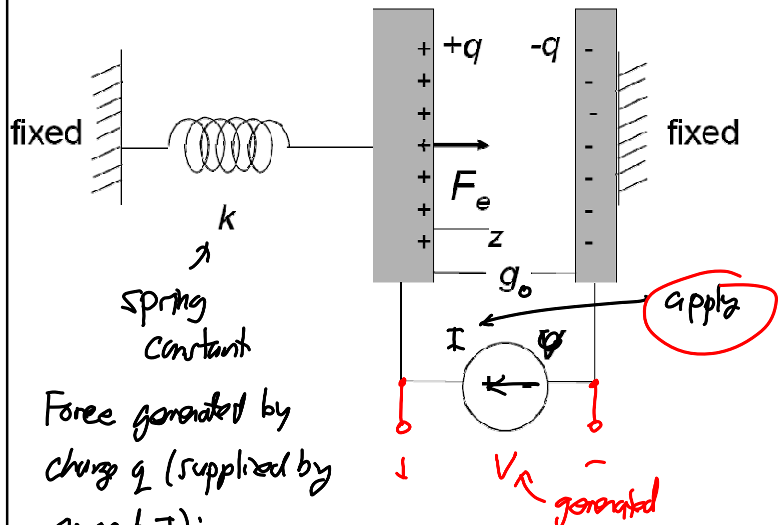
- Reading: Senturia, Chpt. 5, Chpt. 6
- Lecture Topics:
  - Energy Conserving Transducers
    - Charge Control
    - Voltage Control
  - Parallel-Plate Capacitive Transducers
    - Linearizing Capacitive Actuators
    - Electrical Stiffness
  - Electrostatic Comb-Drive
    - 1st Order Analysis
    - 2nd Order Analysis
- ------------------------------------
- Last Time:
- Determined pull-in voltage

over

Charge-Control of a Spring-Suspended C



Force generated by charge q (supplied by current I):

$$F_e = \left.\frac{\partial W(q,g)}{\partial g}\right|_q = \frac{q^2}{2\epsilon A}$$

Restoring force of spring: $F_{spring} = kz = F_e$ ← equilibrium

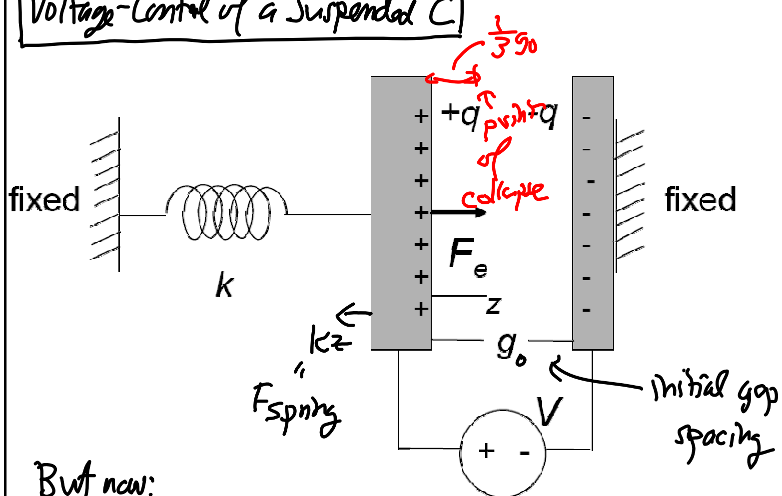
The gap:

$$g = g_o - z = g_o - \frac{F_e}{k} = \boxed{g_o - \frac{1}{2}\frac{q^2}{\epsilon A}\frac{1}{k} = g}$$

↳ $q\uparrow$ can drive $g \to 0$ in continuous fashion

$$V = \frac{q}{C} = \frac{q}{\epsilon A} g = \boxed{\frac{q}{\epsilon A}\left(g_o - \frac{1}{2}\frac{q^2}{\epsilon A}\frac{1}{k}\right) = V}$$ ← $V\downarrow$ as $g\downarrow$

Voltage-Control of a Suspended C



But now:

$$F_e = \frac{\partial W'(V,g)}{\partial g}\Big|_q \rightarrow F_e = \frac{1}{2}\frac{\epsilon A}{g^2}V^2$$

And the gap:

$$g = g_o - z = g_o - \frac{F_e}{k} = \boxed{g_o - \frac{1}{2}\frac{\epsilon A}{g^2}\frac{V^2}{k} = g}$$

g shows up on both sides!

If $V\uparrow \rightarrow g\downarrow \rightarrow F_e\uparrow$

(+) Feedback!

If loop gain > 1, then this will go unstable!

plate will collapse! (into the electrode)

Charge: (for a stable gap)

$$q = \frac{\partial W'(V,g)}{\partial V}\Big|_g = CV \checkmark \quad \text{(as expected)}$$

Stability Analysis

⇒ determine under what conditions voltage control will cause collapse of the plates:

$$F_{net} = F_e - F_{spring} = \underbrace{\frac{\epsilon A V^2}{2g^2}}_{F_e} - \underbrace{k(g_o - g)}_{spring}$$

What happens when I change g by a small increment dg?

↳ get an increment in the net attractive force F_{net}

$$\underbrace{dF_{net}}_{(-)} = \frac{\partial F_{net}}{\partial g}dg = \left[-\frac{\epsilon A V^2}{g^3} + k\right]\underbrace{dg}_{(-)}$$

If $g\downarrow \rightarrow dg = (-)$, then for stability need $F_{net}\downarrow \rightarrow dF_{net} = (-)$

This must be (+)! → otherwise, the plates collapse!

Thus:

$$\boxed{k > \frac{\epsilon A V^2}{g^3}}$$ (for a stable uncollapsed system)

Pull-in Voltage V_{PI} & Pull-in Gap g_{PI}

$V_{PI} \triangleq$ voltage @ which plates collapse

$g_{PI} \triangleq$ gap @ " " "

The plates go unstable when:

$$k = \frac{\epsilon A V_{PI}^2}{g_{PI}^3} \qquad (1)$$

$$F_{net} = 0 = \frac{\epsilon A V_{PI}^2}{2 g_{PI}^2} - k(g_o - g_{PI}) \qquad (2)$$

Substitute (1) into (2):

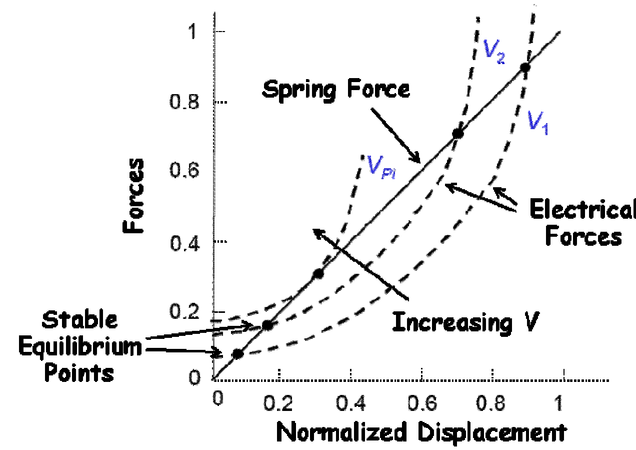
$$0 = \frac{\cancel{\epsilon A V_{PI}^2}}{2\cancel{g_{PI}^2}} - \frac{\cancel{\epsilon A V_{PI}^2}}{g_{PI}^{\cancel{3}}}(g_o - g_{PI})$$

$$\frac{g_o - g_{PI}}{g_{PI}} = \frac{1}{2} \rightarrow g_o = \frac{3}{2} g_{PI}$$

$$\therefore \boxed{g_{PI} = \frac{2}{3} g_o}$$

When the gap is driven by a voltage to (2/3) the initial gap → collapse!

$$V_{PI} = \sqrt{\frac{k g_{PI}^3}{\epsilon A}} \rightarrow \boxed{V_{PI} = \sqrt{\frac{8}{27}\frac{k g_o^3}{\epsilon A}}}$$



Advantages of Electrostatic Actuators:

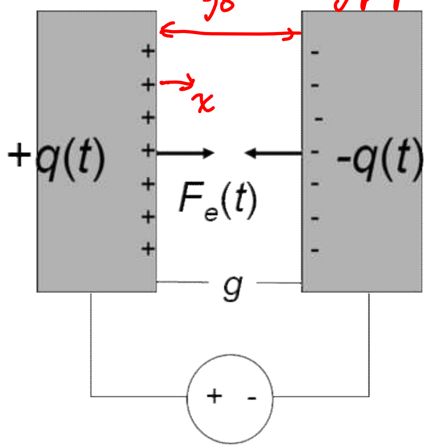
- Easy to manufacture in micromachining processes, since conductors and air gaps are all that's needed → low cost!
- Energy conserving → only parasitic energy loss through I^2R losses in conductors and interconnects
- Variety of geometries available that allow tailoring of the relationships between voltage, force, and displacement
- Electrostatic forces can become very large when dimensions shrink → electrostatics scales well!
- Same capacitive structures can be used for both drive and sense of velocity or displacement
- Simplicity of transducer greatly reduces mechanical energy losses, allowing the highest Q's for resonant structures

Disadvantages of Electrostatic Actuators:

- Nonlinear voltage-to-force transfer function
- Relatively weak compared with other transducers (e.g., piezoelectric), but things get better as dimensions scale

$\boxed{\text{Linearizing the Voltage-to-Force Transfer Fcn}}$

g_0 = initial gap spacing



$$v(t) = \underbrace{V_P}_{\text{DC-Bias}} + \underbrace{v_i(t)}_{\text{signal (AC)}}$$

$$F_e(t) = \frac{\partial W'}{\partial x} = \frac{\partial}{\partial x}\left[\frac{1}{2}C[v(t)]^2\right]$$

$$= \frac{1}{2}\frac{\partial C}{\partial x}[v(t)]^2 = \frac{1}{2}\frac{\partial C}{\partial x}[V_P + v_i(t)]^2$$

$$= \frac{1}{2}\left[V_P^2 + 2V_P v_i(t) + \cancel{[v_i(t)]^2}\right]\frac{\partial C}{\partial x}$$

$$[V_P \gg v_i(t)] \Rightarrow \underbrace{\frac{1}{2}V_P^2\frac{\partial C}{\partial x}}_{\text{DC Offset}} + \underbrace{V_P\frac{\partial C}{\partial x}v_i(t)}_{\text{AC Drive Signal}}$$

$$C_o = \frac{\epsilon A}{g_o} \rightarrow C(x) = \frac{\epsilon A}{g_o - x} = C_o\left(1 - \frac{x}{g_o}\right)^{-1}$$

binomial theorem

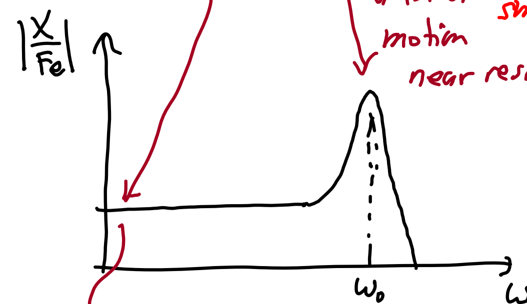
$$[x \ll g_o] \Rightarrow \approx C_o\left(1 + \frac{x}{g_o}\right)$$

$$\therefore \frac{\partial C}{\partial x} = \frac{C_o}{g_o} = \frac{\epsilon A}{g_o^2}$$

$$\Rightarrow F_e(t) = \underbrace{\frac{1}{2}\frac{C_o}{g_o}V_P^2}_{\text{DC Offset}} + V_P\underbrace{\frac{C_o}{g_o}}_{\sim\text{constant for small amplitudes}}v_i(t)$$

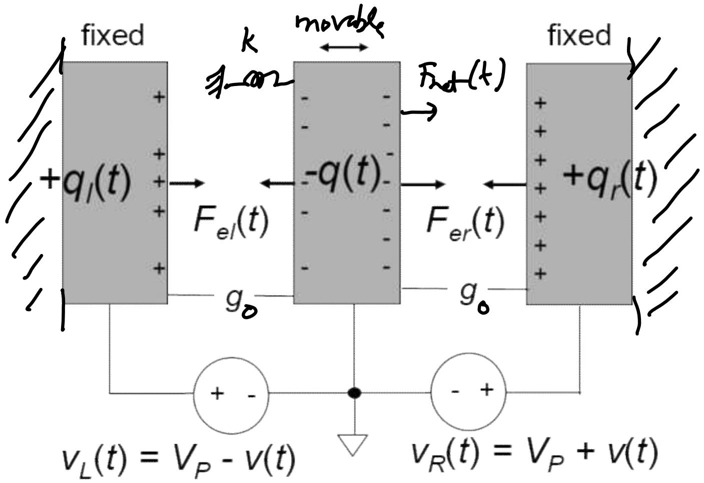
$\therefore$ this is a linear dependence!
↓
only holds for small amplitudes

$v_i \ll V_P$ can still instigate a lot of motion near resonance!



very small response
↳ But still must worry about pull-in, V_{PI}!

Can Cancel the DC Offset Via Differential Symmetry



$$F_{net}(t) = F_{er}(t) - F_{el}(t)$$

$$= \frac{1}{2}\frac{\partial C}{\partial x}\left\{[v_R(t)]^2 - [v_L(t)]^2\right\}$$

$$= \frac{1}{2}\frac{\partial C}{\partial x}\left\{\cancel{V_P^2} + 2V_P v(t) + \cancel{[v(t)]^2} - \left(\cancel{V_P^2} - 2V_P v(t) + \cancel{[v(t)]^2}\right)\right\}$$

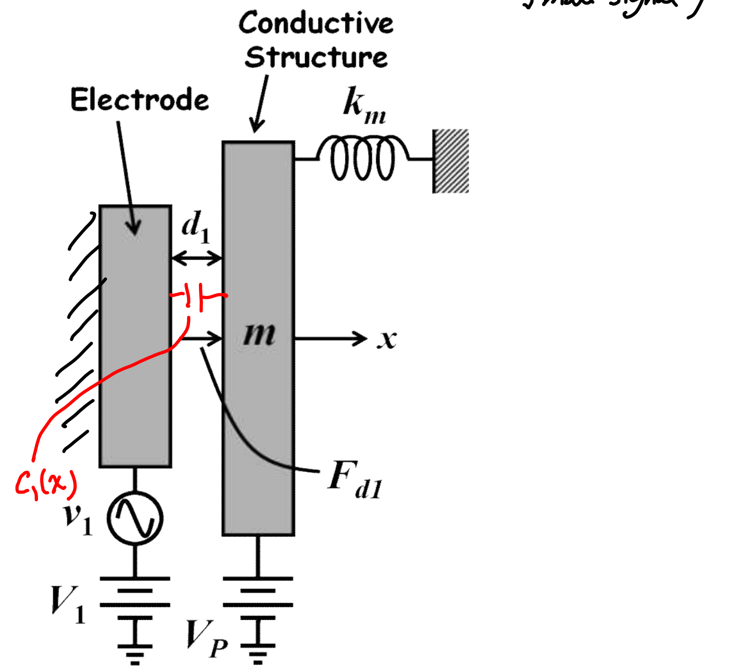
$$\therefore \boxed{F_{net}(t) = 2V_P\frac{\partial C}{\partial x}v(t) = 2V_P\frac{C_o}{g_o}v(t)}$$

quite linear w/ $v(t)$!

still an approximation to $\frac{\partial C}{\partial x}$!

Nonlinearity Still Effects Us

(even though we're using small signals)



More Complete Expressions

$$C_1(x) = \frac{\epsilon A}{d_1 + x} = C_{01}\left(1 + \frac{x}{d_1}\right)^{-1} \rightarrow \frac{\partial C_1}{\partial x} = -\frac{C_{01}}{d_1}\left(1 + \frac{x}{d_1}\right)^{-2}$$

[Expand into Taylor Series]

$$\frac{\partial C_1}{\partial x} = -\frac{C_{01}}{d_1}\left(1 + A_1 x + A_2 x^2 + A_3 x^3 + \cdots\right)$$

where $A_1 = -\frac{2}{d_1}$, $A_2 = \frac{3}{d_1^2}$, $A_3 = -\frac{4}{d_1^3}$, ...

$$F_{d1} = \frac{1}{2}\frac{\partial C_1}{\partial x}(\underbrace{V_P - V_1}_{} - v_1)^2 = \frac{1}{2}\frac{\partial C}{\partial x}(V_{P1} - v_1)^2$$

$$V_{P1} = V_P - V_1$$

[small displacements: $x \ll d_1$]

$$\frac{1}{2}(1 + \cos 2\omega_o t) = \cos^2\omega_o t$$

$$F_{d1} = \frac{1}{2}\left(-\frac{C_{o1}}{d_1}\right)(1 + A_1 x)(V_{P1}^2 + 2V_{P1}v_1 + v_1^2)$$

$$v_1 = V_1 \cos\omega_o t$$

$$= \frac{1}{2}\left(-\frac{C_o}{d_o}\right)\{V_{P1}^2 + 2V_{P1}v_1 + v_1^2 + A_1 V_{P1}^2 x - 2A_1 V_{P1} x v_1 + A_1 x v_1^2\}$$

$\rightarrow \omega_o$

Resonance: $\left|\frac{X}{F_d}\right|$