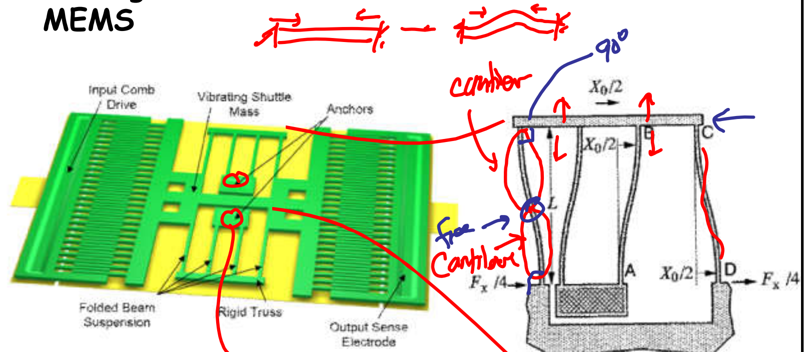
Lecture 13: Beam Bending

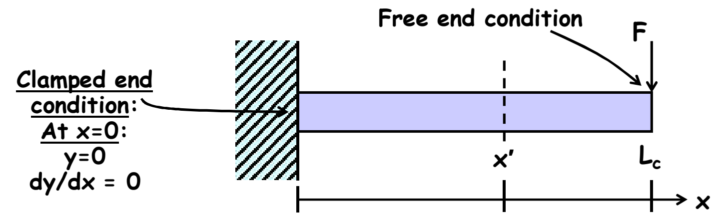
- Announcements:
- HW#4 online soon
- Module 8 on "Microstructural Elements" online
- Graded HW#1 handed back
- -----------------------------------
- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients
- -----------------------------------
- Last Time:
- Started Bending of Beams
- Continue with this …

- Springs and suspensions very common in MEMS
- Coils are popular in the macro-world; but not easy to make in the micro-world
- Beams: simpler to fabricate and analyze; become "stronger" on the micro-scale → use beams for MEMS



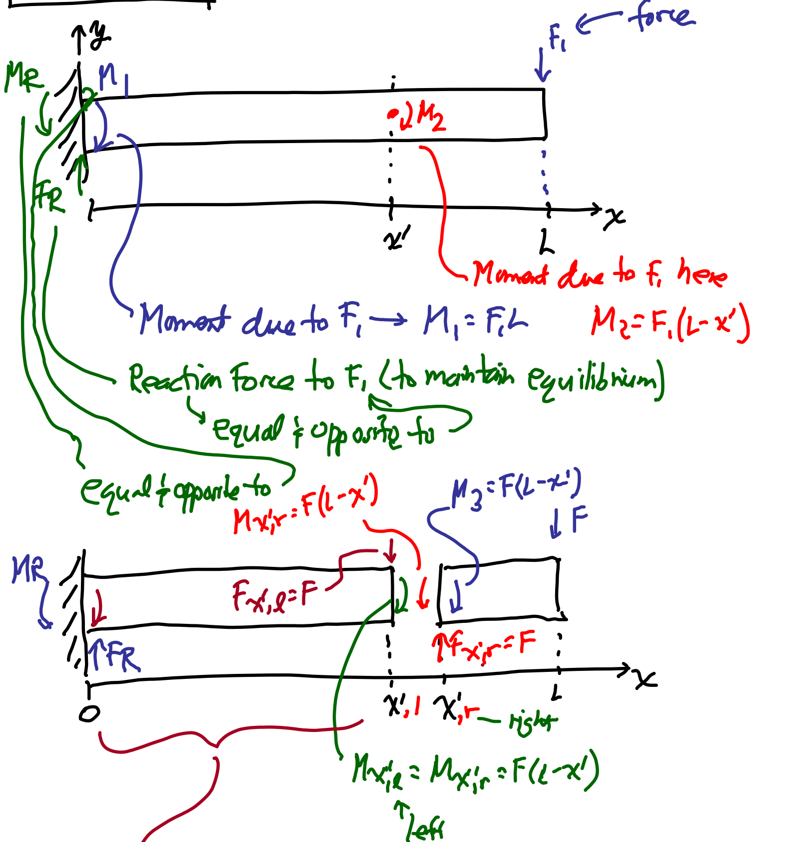
Comb-Driven Folded Beam Actuator

Problem: Bending a Cantilever Beam



- Objective: Find relation between tip deflection $y(x=L_c)$ and applied load F
- Assumptions:
  1. Tip deflection is small compared with beam length
  2. Plane sections (normal to beam's axis) remain plane and normal during bending, i.e., "pure bending"
  3. Shear stresses are negligible

Forces & Moments

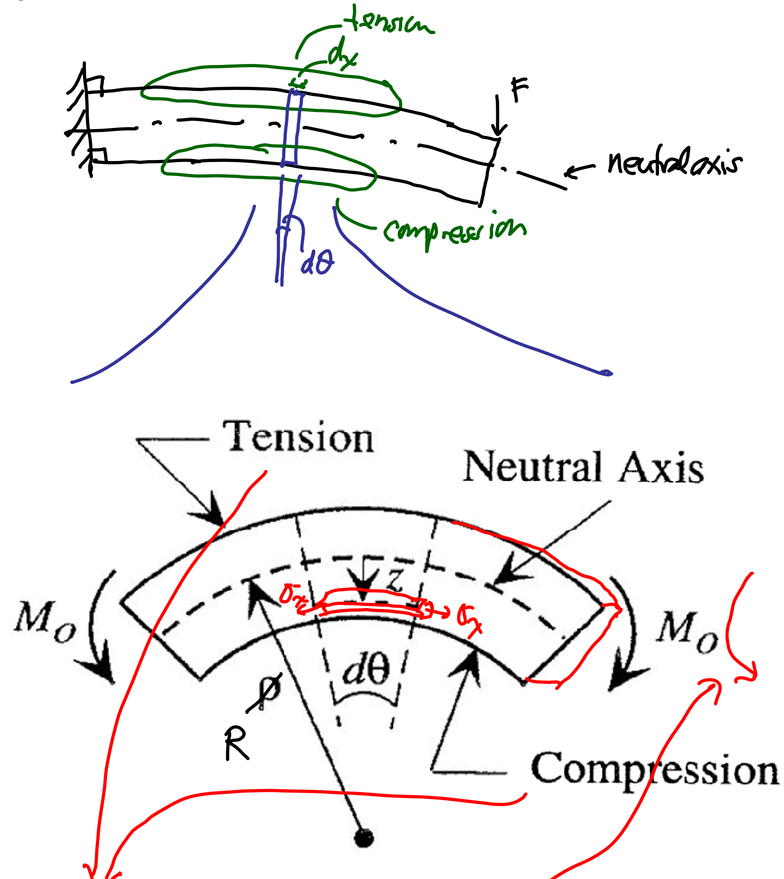


Total Force: $-F_R + F_{x',l} = -F + F = 0$ ✓

Total Moment: $-M_R + M_{x',l} + x' F_{x',l}$

$= -FL + F(L - x') + x'F = 0$ ✓

Bent Beam (bent by applied Force)



Combine to generate forces → moments that fight against the moment due to F

↓

This is how stiffness arises!

↓

k_c → we want this!

Beam Segment in Pure Bending

⇒ consider the segment bounded by the dashed lines defining $d\theta$

At $z=0$: neutral axis → segment length $= dx = R\,d\theta$ (1)

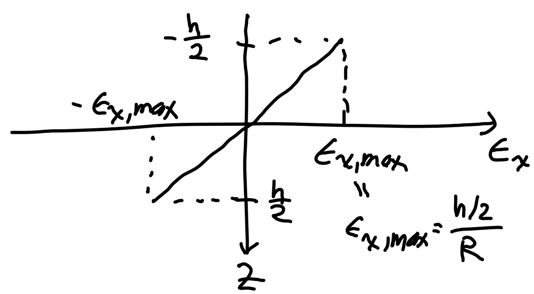
At any z: segment length $: dL = (R-z)\,d\theta$ (2)

Combine (1) & (2): $dL = dx - z\,d\theta = dx - \frac{z}{R}dx$

Thus, the axial strain @ z: $\epsilon_x = \frac{dL-dx}{dx} = -\frac{z}{R}$

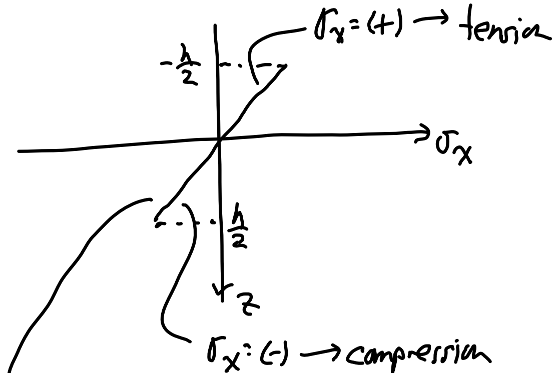
$$\boxed{\epsilon_x = -\frac{z}{R}}$$

Thus, the strain varies linearly along the beam thickness!



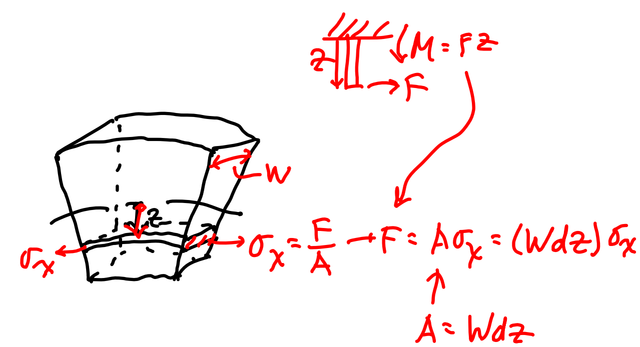
Of course, there is a corresponding axial stress.

$$\sigma_x = \epsilon_x E = \boxed{-\frac{zE}{R} = \sigma_x}$$



This gradient of stress then generates a bending moment! in response to the original applied moment (from F)

Stress → Force:



⇒ integrate stress through the beam thickness:

$$M = \int_{-\frac{h}{2}}^{\frac{h}{2}} [\underbrace{(W dz)\sigma_x}_{\text{force}}] \cdot z$$

$$= \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{EWz^2}{R} dz \Rightarrow \boxed{M = -\left(\frac{1}{12}Wh^3\right)\frac{E}{R}}$$

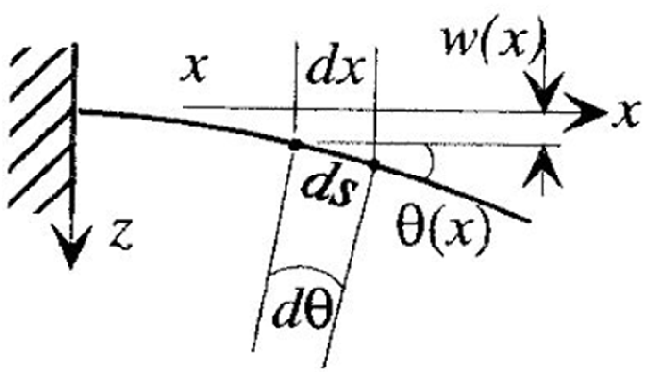
$$\left[\sigma_x = -\frac{zE}{R}\right]$$

$\frac{1}{12}Wh^3 = I \triangleq$ Moment of Inertia

$$\boxed{\frac{1}{R} = -\frac{M}{EI}}$$

Note: (+) radius of curvature
↓
(−) internal bending moment

Differential Equation for Beam Bending



Write out some geometric relationships:

→ then use small angle approximation

$$\cos\theta = \frac{dx}{ds} \rightarrow ds = \frac{dx}{\cos\theta} \longrightarrow ds = dx$$

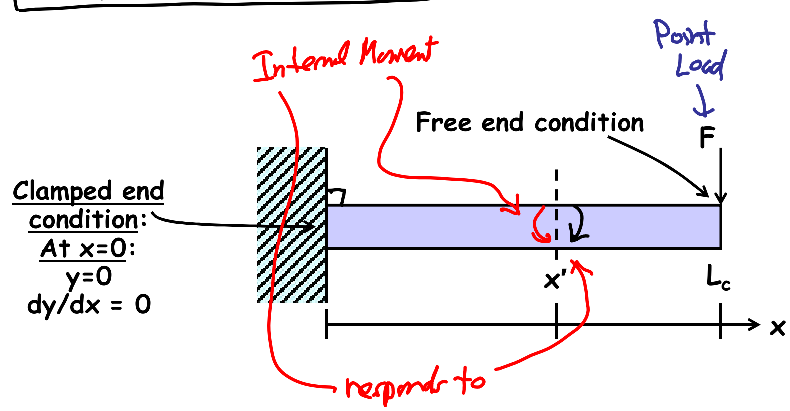
$$\tan\theta = \frac{dw}{dx} = \text{slope of the beam at this point} \longrightarrow \theta \cong \frac{dw}{dx} \quad (1)$$

$$ds = R\,d\theta \rightarrow \frac{1}{R} = \frac{d\theta}{ds} \longrightarrow \frac{1}{R} = \frac{d\theta}{ds} \quad (2)$$

Inserting (1) into (2):

$$\frac{1}{R} = \boxed{\frac{d^2w}{dx^2} = -\frac{M}{EI}}$$ ← Diff. Eq. for Small Angle Beam Bending

Cantilever Beam w/ Concentrated Load



Internal Moment @ position x: $M = -F(L-x)$

Thus: $$\frac{d^2w}{dx^2} = \frac{F}{EI}(L-x)$$

w/ Clamped End B.C.: $w(x=0)=0$, $\frac{dw}{dx}(x=0)=0$

Free End B.C.: none

Solve to get w:

⇒ use Laplace; a trial solution

$w = A + Bx + Cx^2 + Dx^3$, then apply B.C.'s

$$\boxed{w = \frac{FL}{2EI}x^2\left(1-\frac{x}{3L}\right)}$$

Deflection @ x due to a point load F applied @ $x=L$

Maximum Deflection → occurs $x=L$:

$$w_{max} = \left(\frac{L^3}{3EI}\right)F \rightarrow F = \underbrace{\left(\frac{3EI}{L^3}\right)}w(x=L) = k_c w(x=L)$$

stiffness ≙ k_c @ $x=L$

$$k_c = \frac{3EI}{L^3} \Rightarrow \boxed{k_c = \frac{1}{4}EW\frac{h^3}{L^3}}$$

$$\left[I = \frac{1}{12}Wh^3\right]$$

Ex. $L = 100\mu m$, $W = 2\mu m$, $h = 2\mu m$

polysilicon → $E = 150$ GPa

$$k_c = \frac{1}{4}(150G)(2\mu)\left(\frac{2\mu}{100\mu}\right)^3 = \underline{0.6\ N/m}$$

Maximum Stress in a Bent Cantilever

From before, the radius of curvature is given by:

$$\frac{1}{R} = \frac{d^2w}{dx^2} = \frac{F}{EI}(L-x)$$

⇒ $\frac{1}{R}$ is maximized (i.e., R is minimized) when $x=0$: $[x=0] \Rightarrow \frac{1}{R} = \frac{d^2w}{dx^2} = \frac{FL}{EI}$

Strain is maximized

① At top surface → tensile

② At bottom surface → compressive

@ $x=0$

$$\epsilon_{max} = \frac{z}{R} = \frac{h}{2}\frac{1}{R} = \boxed{\frac{h}{2}\frac{FL}{EI} = \epsilon_{max}}$$

$$\left[I = \frac{1}{12}Wh^3\right] \Rightarrow \epsilon_{max} = \frac{\cancel{h}}{2}\frac{FL}{E}\left(\frac{12}{Wh^{\cancel{3}2}}\right) = \frac{6L}{EWh^2}F$$

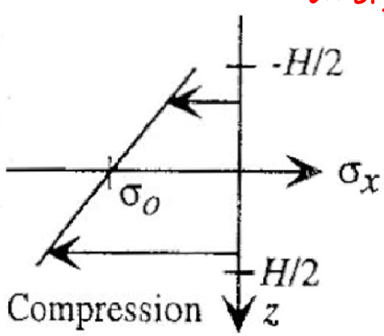
$$\boxed{\sigma_{max} = \epsilon_{max} E = \frac{6L}{Wh^2}F}$$

(maximum stress in a bent cantilever subjected to a force F at its tip)

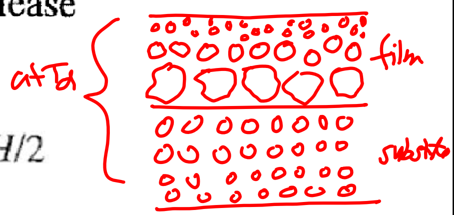
$\boxed{\text{Stress Gradient in Cantilevers}}$

① Deposit film @ high temperature, T_d

Before release



Stress before release



② Cool it down.