Lecture 15: Beam Combos II

- Announcements:
  - HW#4 online, due Tuesday, next week, 10 a.m.
    - Extended from last time
  - Midterm Exam, Thursday, March 22, 11-12:30 p.m., 3109 Etcheverry (right here)

- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients

- Last Time:
  - Spring circuits ... continue with this

Typical Questions: → all demand that we know \( x = f(\epsilon) \)

- How does the structure move in response to a force at a specific location?
- What is the frequency response to an AC force applied at a specific location?
- Noise?
- Response to environmental stimuli? (e.g., vibration)
- How does stress affect the behavior of the structure?
Procedure:

1. Build the circuit. (Extract-like circuit) → in the x-direction
   (for this example)

2. Analyze to get x = f(F) = force
   displacement

(a) Case 1: Series
   - k_1 x_1 = k_2 x_2
   - x = \frac{F}{k}

(b) Case 2: Parallel Springs

k_{tot} = k_1 \parallel k_2

\begin{align*}
    k_{tot} &= k_1 \parallel k_2 \\
    &= \frac{k_1 k_2}{k_1 + k_2}
\end{align*}

For EE circuits, springs combine like capacitors

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\[ C_1 \parallel C_2 = C \]

\[ F = k_1 x_1 = k_2 x_2 \]

\[ F = \frac{1}{C_{tot}} = \frac{1}{C_1 + C_2} \]

or only need to go through one of the springs to get from the circuit to the forcing pt.

\[ F = F_1 + F_2 = (k_1 + k_2) x_{tot} \]

\[ k_{tot} = k_1 + k_2 \]

(For k_1 \parallel k_2 in parallel)
Series Combination of Springs

[(Fixed B.C.)] \rightarrow \text{Anchor} \rightarrow \text{Free (just like a cantilever)} \rightarrow \text{Cantilever}\#1 \rightarrow \text{Cantilever}\#2 \rightarrow \text{Cantilever B.C. (maintain y=0)}

\[ y(L) \]

\[ y_{\text{tot}} = y_1 + y_2 \rightarrow \text{series}\]

\[ Y_{\text{tot}} \text{ must go thru both springs} \rightarrow \text{series}\]

\[ k_{\text{tot}} = k_1 k_2 \cdot \frac{1}{k_1 + k_2} = \frac{k_1 k_2}{k_1 + k_2} \]

\[ \text{Cantilever stiffness} \]

Parallel Combination of Beams

\[ F/2 \]

\[ F = \text{forcing pt.} \]

\[ Y_{\text{tot}} = Y_a = Y_b \rightarrow \text{Parallel} \]

\[ k_{\text{tot}} = k_a + k_b \]

\[ \text{To go from anchor to forcing pt., need only go through one of the beams} \]
**Stiffness of Folded-Beam Suspension**

- Parallel
- Series

- \( K_c = \frac{3EIz}{L_c} = \frac{1}{4} Eh\left(\frac{w}{L_c}\right)^3 \)
- Invert \( L_c = \frac{L}{2} \)

- \( k_{bc} = \frac{3EIz}{(L/2)^3} = \frac{24EIz}{L^3} \)
- Full beam length

**Micromechanical Fill**

- Find the stiffness at point A.
- Apply \( F \) at A. What is \( x_A \)?
- \( x_A = \frac{F}{k_{bc}} \)
- (Shutters + folding trusses are rigid)

**Wrench**

- Apply forces to point B.

**Anchor**

- Mass of shuttle 1:
- Mass of shuttle 2

**Get** \( k_{bc} = \frac{KcI}{2} \)
Get \( k_b \):

\[ k_b = k_{cs} - \frac{k_{cs}}{4} \]

Series

\[ k_{cs} \]

Parallel

\[ \frac{k_{cs}}{2} \]

\[ k_A = k_c + k_{comb} = k_c + k_{ll}l k_{ll} \frac{k_{cs}}{2} = k_A \]

where

\[ k_c = \frac{2yE}{L^3} \]

\[ k_{cs} = \frac{2yE}{L^3} \]

Tensioned Spring (Non-Ideality)

- Important case for MEMS suspensions, since the thin films comprising them are often under residual stress.
- Consider small deflection case: \( y(x) \ll L \)

Governing differential equation: (Euler Beam Equation)

\[ EI \frac{d^4y}{dx^4} - S \frac{d^2y}{dx^2} = F \delta(x - L) \]

Axial Load

Unit impulse @ \( x=L \)

Heuristic Derivation for the Euler Beam Equation

Consider first a straight beam under axial stress:

\[ \sigma_c \]

\[ \rightarrow \sigma_0 \]

\[ \rightarrow \text{no effect on } z\text{-directed stiffness when the beam is straight} \]

...but when the beam is bent:

Thin beam

Axial Stress

2-directed component

\( P \)

\[ 0 \text{WH} \]

\[ 0 \text{WH} \]
* Upward pressure $P_o$ to counteract the downward force from the beam to keep everything in static equilibrium.

For ease of analysis:

Assume the beam is bent to an angle $\theta$.

Downward vertical force: $25\text{WHT}$

Upward force due to $P_o$:

$$P_y(x) = P_o \sin \theta$$

$$F_u = \int_0^L (P_o \sin \theta) W(x) dx$$

$$= -P_o W \cos \theta \bigg|_0^L$$

$$= 2R P_o$$

[Equilibrium] $2R P_o = 25\text{WHT} \rightarrow P_o \frac{50}{R}$

$$q_x = \text{beam load/unit length} = P_o W, \quad \frac{d^2 w}{dx^2}$$

beam displacement

$$q_x = 50 \text{WHT} \frac{d^2 w}{dx^2}$$

generalizes to the case of small displacement and angles.

Now use the differential beam bending equation:

$$\frac{d^2 w}{dx^2} = -\frac{M}{EI}$$

$$\frac{d^4 w}{dx^4} = \frac{q}{EI}$$

* Relationship between forces on a fully-loaded beam element.

$q_x = \text{force/unit length}$