Lecture 15: Beam Combos II

- **Announcements:**
- HW#4 online, due Tuesday, next week, 10 a.m.
  - Extended from last time
- Midterm Exam, Thursday, March 22, 11-12:30 p.m., 3109 Etcheverry (right here)
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- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients
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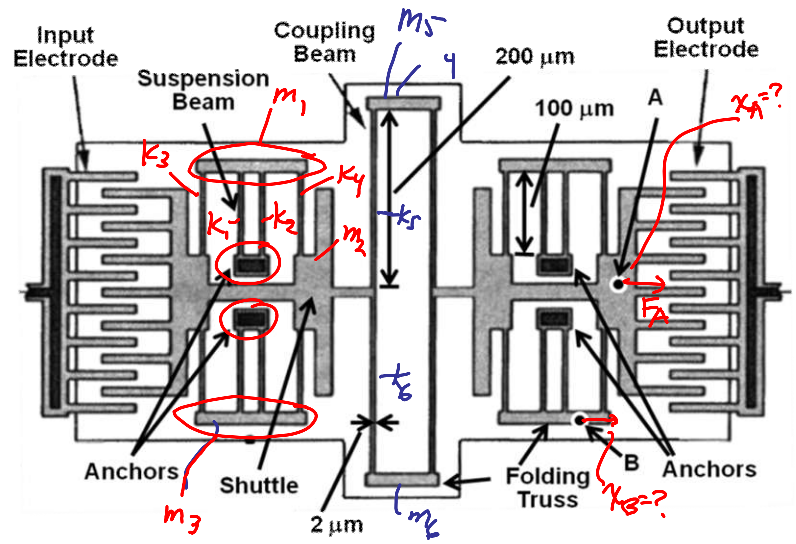
- **Last Time:**
- Spring circuits … continue with this

over

Analyzing an Interconnected Ensemble of Beams (Springs) & Masses



Typical Questions: → all demand that we know $x = f(F)$

① How does the structure move in response to a force at a specific location. ↪ = that we know

② What is the frequency response to an AC force applied at a specific location. ↪ the stiffness?

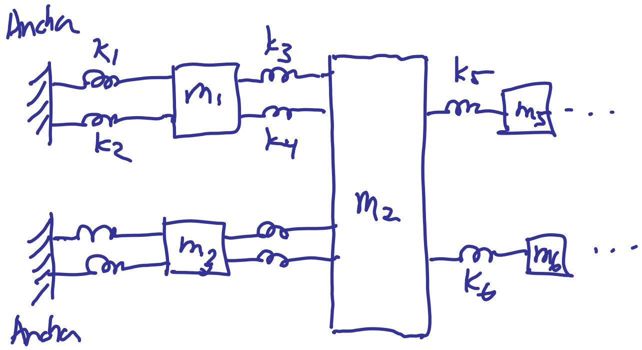
③ Noise?

④ Response to environmental stimuli? (e.g., rotation)

⑤ How does stress affect the behavior of the structure?

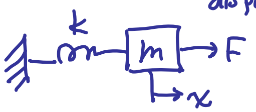
Procedure:

① Build the ckt. (Extract the ckt.) → in the x-direction (for this example)



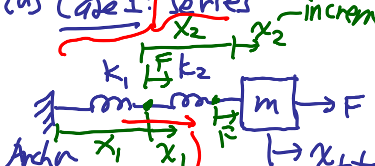
② Analyze to get $x = f(F)$

x ← displacement; F ← force



$\Rightarrow F = kx \rightarrow x = \frac{F}{k} = \left(\frac{1}{k}\right)F$ ← compliance

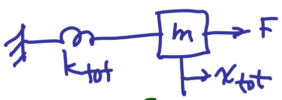
(a) Case 1: Series — series because one must go through both k_1 & k_2 to get from the anchor to the forcing pt.



x_{tot} ← want this ∴ need k_{tot}

x → across variable; F → thru variable

$x_{tot} = \left(\frac{1}{k_{tot}}\right)F$



$x_1 = \frac{F}{k_1} \quad x_2 = \frac{F}{k_2}$

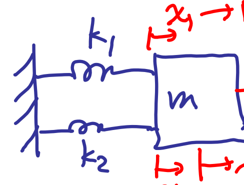
$\underbrace{x_{tot} = x_1 + x_2}_{\text{identifies series.}} = F\left(\frac{1}{k_1} + \frac{1}{k_2}\right) = \frac{F}{(k_1 \| k_2)} = \frac{F}{k_{tot}}$ *

* $\left[\text{"}\|\text{" operator} \triangleq A\|B = \frac{1}{\frac{1}{A}+\frac{1}{B}} = \frac{AB}{A+B}\right]$ $C_1 \| C_2$

$\boxed{k_{tot} = k_1 \| k_2}$ (for k_1 & k_2 in series) C_1 C_2

For EE's: springs combine like capacitors

(b) Case 2: Parallel Springs



$x_1 \rightarrow F_1 = k_1 x_{tot}$

$F = k_{tot} x_{tot}$ ← want k_{tot}

C_1 C_2 $\} = $ $C_1 + C_2$

$x_2 \rightarrow F_2 = k_2 x_{tot}$

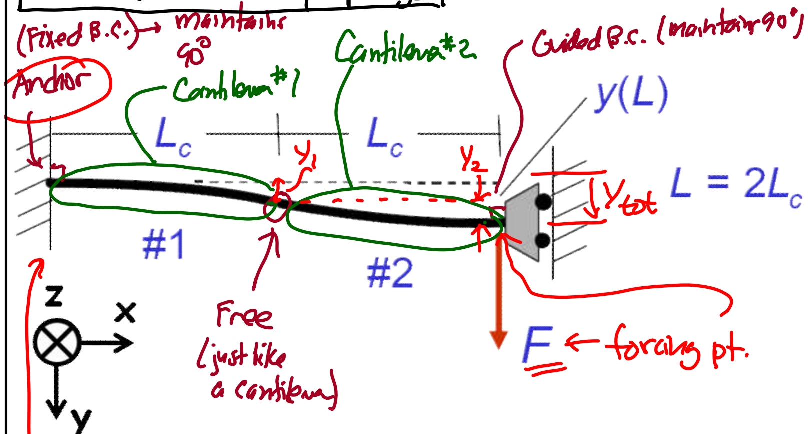
$\underbrace{x_{tot} = x_1 = x_2}_{\text{indicates parallel}}$

or → only need to go through one of the springs to get from the anchor to the forcing pt.

$F = F_1 + F_2 = \underbrace{(k_1 + k_2)}_{k_{tot}} x_{tot}$

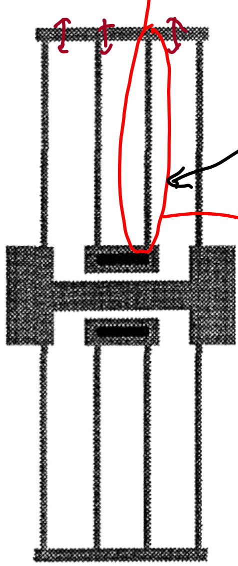
$\boxed{k_{tot} = k_1 + k_2}$ (for k_1 & k_2 in parallel)

Series Combination of Springs



$y_{tot} = y_1 + y_2$ ∴ series ✓

must go thru both springs to get from anchor to forcing pt. ⇒ series ✓

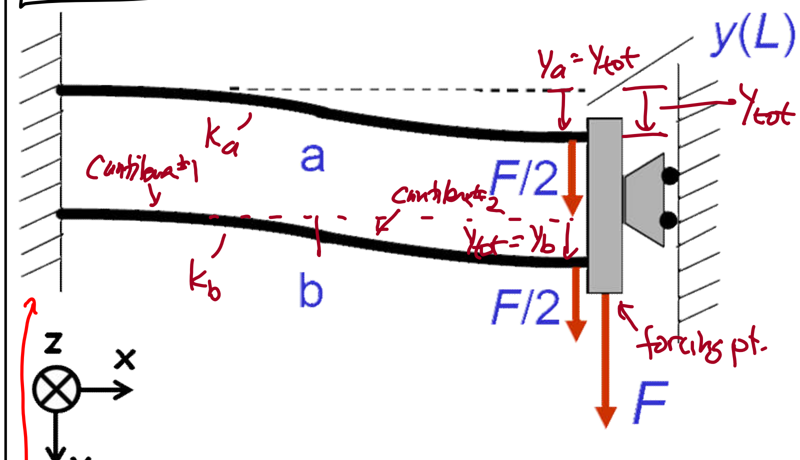


$$k_{tot} = k_1 \| k_2 = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}} = \frac{k_1 k_2}{k_1 + k_2}$$

$(k_1 = k_2 = k_c) \Rightarrow k_{tot} = \frac{k_c}{2}$

cantilever stiffness

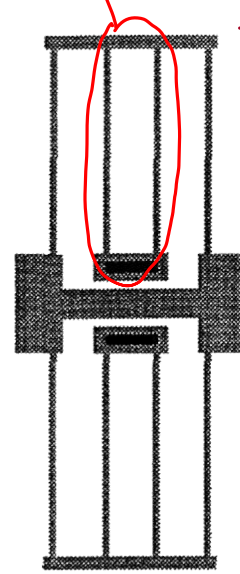
Parallel Combination of Beams



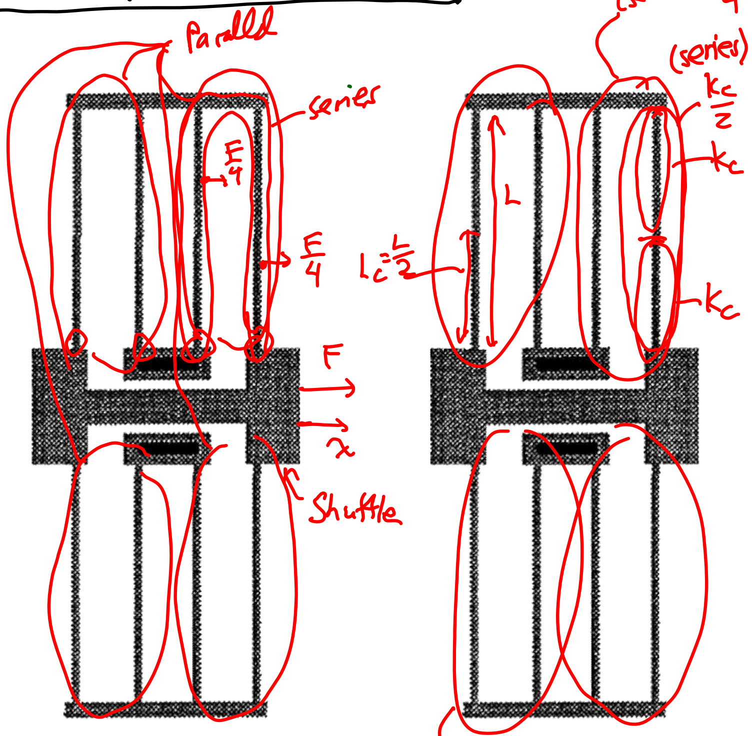
$y_{tot} = y_a = y_a$ → Parallel

To go from anchor to forcing pt., need only go through one of the beams

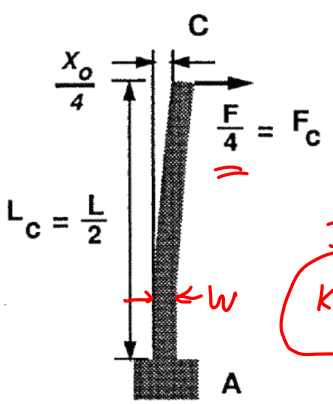
$$k_{tot} = k_a + k_b$$



Stiffness of Folded-Beam Suspension



$k_{tot} = 4\left(\frac{k_c}{4}\right) = k_c$



$$\boxed{k_c = \frac{3EI_z}{L_c^3} = \frac{1}{4}Eh\left(\frac{W}{L_c}\right)^3}$$

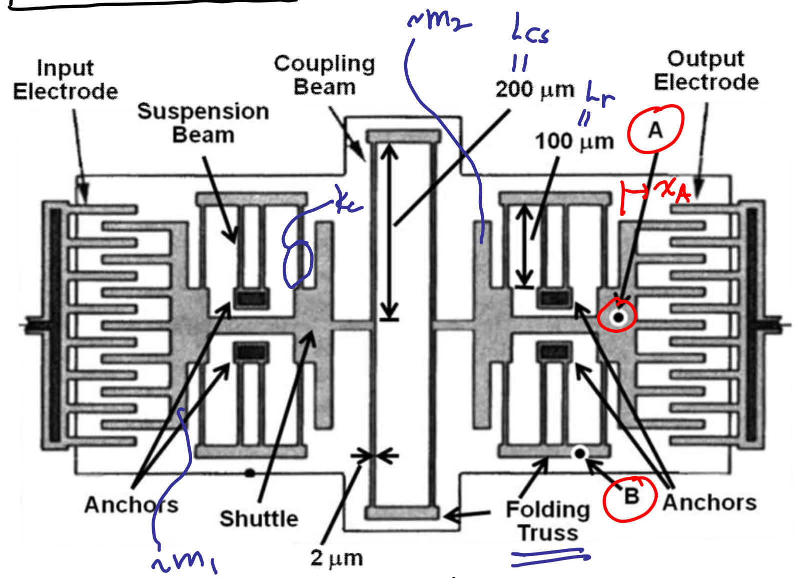
$I_z = \frac{1}{12}hW^3$

Insert $L_c = \left(\frac{L}{2}\right)$

$$\boxed{k_c = \frac{3EI_z}{(L/2)^3} = \frac{24EI_z}{L^3}}$$

full beam length

Micromechanical Filter



⇒ Find the stiffness at point A.

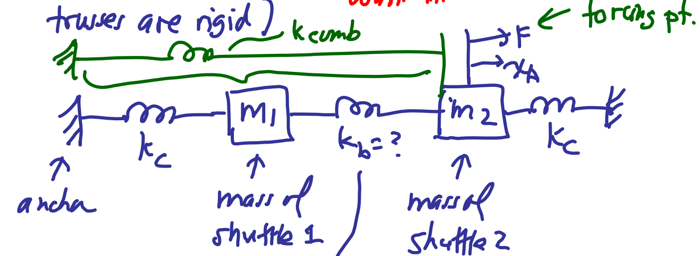
↳ apply F @ A → what is x_A?

$k_A \triangleq$ stiffness @ point A

↖ want this

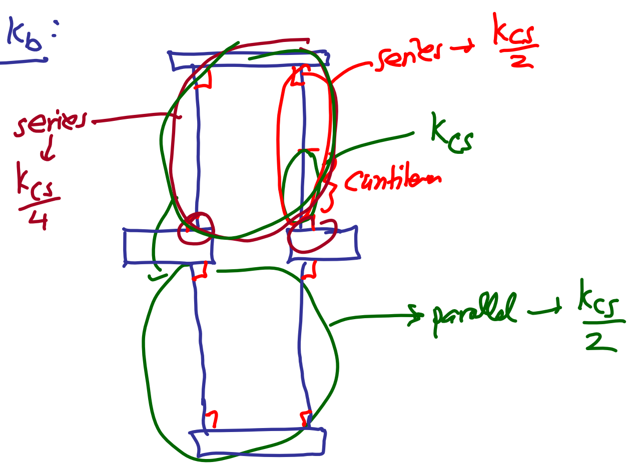
$x_A = \frac{F}{k_A}$

(shuttles & folding trusses are rigid)



Get $k_b = \frac{k_{cs}}{2}$

Get k_b:



$\therefore k_A = k_c + k_{comb}$

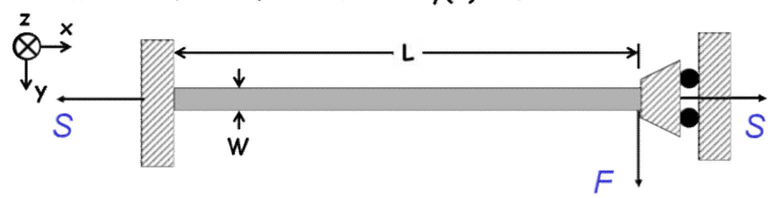
$= k_c + k_c \| k_b$: $k_c + k_c \| \frac{k_{cs}}{2} = k_A$

where $k_c = \frac{24EI_z}{L_r^3}$

$k_{cs} = \frac{24EI_z}{L_{cs}^3}$

Tensioned Spring (Non-Ideality)

- Important case for MEMS suspensions, since the thin films comprising them are often under residual stress
- Consider small deflection case: $y(x) \ll L$



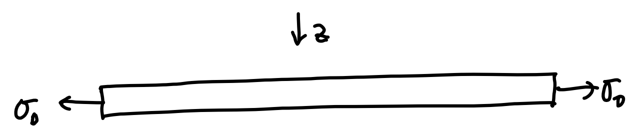
Governing differential equation: (Euler Beam Equation)

$$EI_z \frac{d^4 y}{dx^4} - S\frac{d^2 y}{dx^2} = \underbrace{F\delta(x-L)}_{\text{Unit impulse @ x=L}}$$

Axial Load

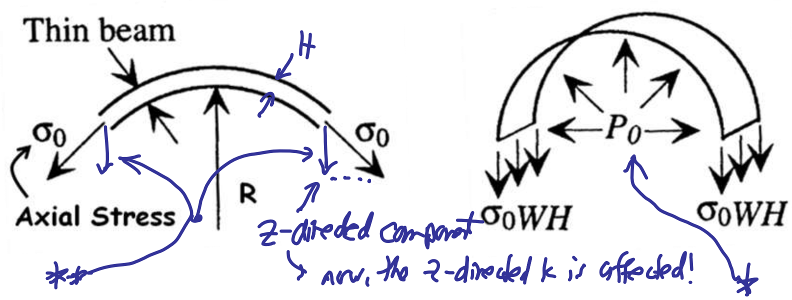
Heuristic Derivation for the Euler Beam Equation

Consider first a straight beam under axial stress:



⟹ no effect on z-directed stiffness when the beam is straight

...but when the beam is bent:



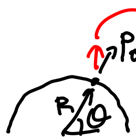
* Upward pressure P_o to counteract the downward force from ** to keep everything in static equilibrium

For ease of analysis:

Assume the beam is bent to an angle π

↳ downward vertical force: $2\sigma_o WH$

Upward Force due to P_o:



$$P_{oy}(\theta) = P_o \sin\theta$$

$$F_u = \int_0^{\pi} (P_o \sin\theta) W (R\,d\theta)$$

$$= -P_o W R \cos\theta \Big|_0^{\pi}$$

$$= 2RWP_o$$

[Equilibrium] $\Rightarrow 2RWP_o = 2\sigma_o WH \rightarrow P_o = \dfrac{\sigma_o H}{R}$

$$\left[q_o = \frac{\text{beam load}}{\text{unit length}} = P_o W, \quad \frac{1}{R} = \frac{d^2 w}{dx^2} \right]$$ beam displacement

$$q_o = \sigma_o WH \frac{d^2 w}{dx^2}$$ → generalizes to the case of small displacements & angles

Now, use the Differential Beam Bending Equation

$$\frac{d^2 w}{dx^2} = -\frac{M}{EI} \rightarrow \frac{d^4 w}{dx^4} = \frac{q}{EI}$$

q ← $\dfrac{\text{load}}{\text{unit length}}$

??? *

* Relationship Between Forces on a Fully-Loaded Beam Element