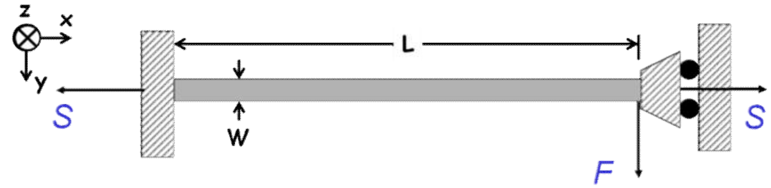
**Lecture 16: Energy Methods**

- Announcements:
- HW#4 online, due Tuesday, next week, 10 a.m.
- Module 9 on "Energy Methods" online
- Midterm Exam 2 weeks away, Thursday, March 22, 11:00-12:30, 3109 Etcheverry (right here)
- ------------------------------------
- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients
- ------------------------------------
- Reading: Senturia, Chpt. 10
- Lecture Topics:
  - Energy Methods
  - Virtual Work
  - Energy Formulations
  - Tapered Beam Example
- ------------------------------------
- Last Time:
- Modeling a tensioned spring
- continue with this …

## Tensioned Spring (Non-Ideality)

- Important case for MEMS suspensions, since the thin films comprising them are often under residual stress
- Consider small deflection case: y(x) « L

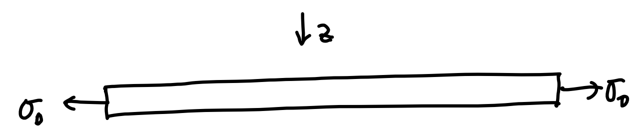



Governing differential equation: (Euler Beam Equation)

$$EI_z\frac{d^4 y}{dx^4} - \underset{\text{Axial Load}}{S}\frac{d^2 y}{dx^2} = \underbrace{F\delta(x-L)}_{\text{Unit impulse @ x=L}}$$

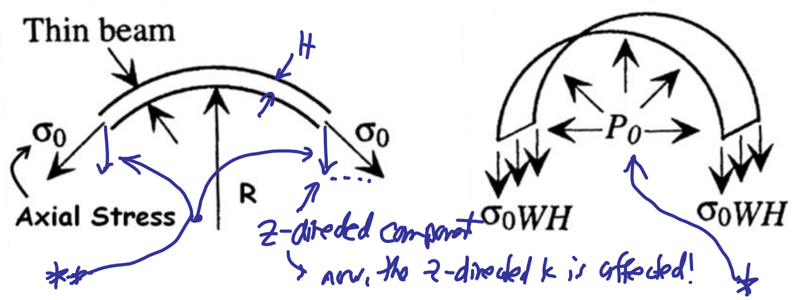
## Heuristic Derivation for the Euler Beam Equation

Consider first a straight beam under axial stress:




⇒ no effect on z-directed stiffness when the beam is straight

...but when the beam is bent:



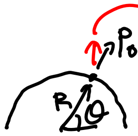
* ↳ Upward pressure $P_o$ to counteract the downward force from ** to keep everything in static equilibrium

For ease of analysis:

Assume the beam is bent to an angle $\pi$

↳ downward vertical force: $2\sigma_o W H$

Upward force due to $P_o$:




$P_{oy}(\theta) = P_o \sin\theta$

$$F_u = \int_0^{\pi} (P_o \sin\theta) W (R\, d\theta)$$

$$= -P_o W R \cos\theta \Big|_0^{\pi}$$

$$= 2RWP_o$$

[Equilibrium] ⇒ $2RWP_o = 2\sigma_o W H \longrightarrow P_o = \frac{\sigma_o H}{R}$

$\left[ q_o = \frac{\text{beam load}}{\text{unit length}} = P_o W, \ \frac{1}{R} = \frac{d^2 w}{dx^2} \right]$ ← beam displacement

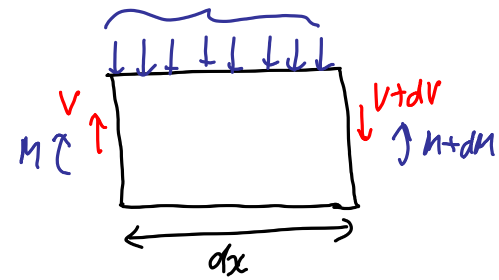
$$q_o = \sigma_o W H \frac{d^2 w}{dx^2}$$ → generalizes to the case of small displacements & angles

Now, use the Differential Beam Bending Equation

$$\frac{d^2 w}{dx^2} = -\frac{M}{EI} \longrightarrow \frac{d^4 w}{dx^2} = \frac{q}{EI}$$ ← load / unit length

??? *

* Relationship Between Forces on a Fully-Loaded Beam Element




[Total Static Equilibrium] ⇒ total force = 0

$$F_T = \text{total force} = q\,dx + (\cancel{V} + dV) - \cancel{V} = 0$$

$$\therefore \boxed{\frac{dV}{dx} = -q} \qquad (1)$$

⇒ also, total moment w/r to the left-hand edge = 0

$$M_T = (\cancel{M} + dM) - \cancel{M} - (V + \cancel{dV})dx - \cancel{\tfrac{1}{2} q\, dx^2} = 0$$

$$\int_0^{dx} (q\,du)u = \tfrac{1}{2} q\, dx^2$$

[neglect products of differentials]

$$dM - V dx = 0 \rightarrow \boxed{\frac{dM}{dx} = V} \qquad (2)$$

Using (1) & (2): $\left[ \frac{d^2 M}{dx^2} = \frac{dV}{dx} = -q \right]$

$$EI \frac{d^4 w}{dx^2} = q + q_o$$

$q$ ← external load; $q_o$ ← equiv. load f/ axial stress

$\left[q_0 = \sigma_0 W H \frac{d^2w}{dx^2}\right] \Rightarrow$

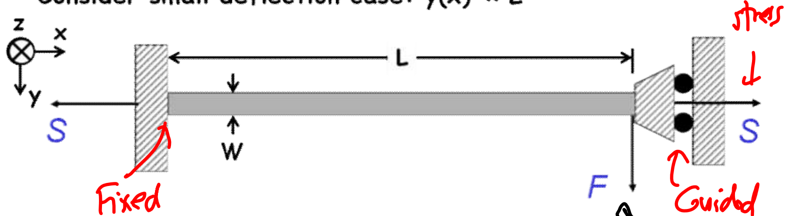
$$\boxed{EI\frac{d^4w}{dx^4} - \underbrace{(\sigma_0 WH)}_{\text{tension in the beam} = S}\frac{d^2w}{dx^2} = q}$$

a force

Euler Beam Equation

## Clamped-Guided Beam Under Axial Load

- Important case for MEMS suspensions, since the thin films comprising them are often under residual stress
- Consider small deflection case: y(x) « L




Governing differential equation: (Euler Beam Equation)

$$EI_z\frac{d^4y}{dx^4} - S\frac{d^2y}{dx^2} = \underbrace{F\delta(x-L)}_{\text{Unit impulse @ x=L}}$$

Force

Axial Load

Need to solve this, then find the stiffness against this force (@ this location)

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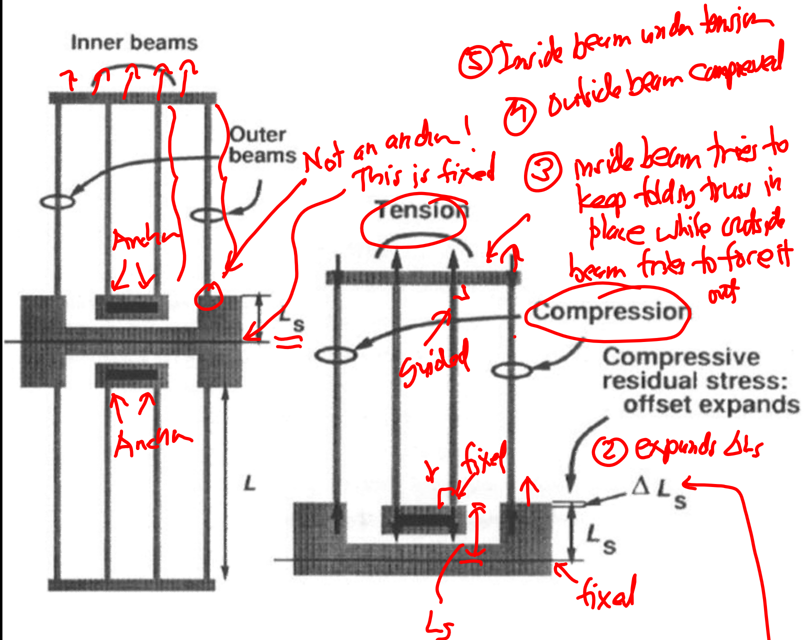
- Can solve the ODE using standard methods
  - Senturia, pp. 232-235: solves ODE for case of point load on a clamped-clamped beam (which defines B.C.'s)
  - For solution to the clamped-guided case: see S. Timoshenko, Strength of Materials II: Advanced Theory and Problems, McGraw-Hill, New York, 3rd Ed., 1955
- Result from Timoshenko:

S > 0 (tension) $\quad k^{-1} = \frac{pL - 2\tanh(pL/2)}{p|S|} = \frac{y(x=L)}{F}$ $\quad k^{-1}_{ten}$

S < 0 (compression) $\quad k^{-1} = \frac{-pL + 2\tan(pL/2)}{p|S|} = \frac{y(x=L)}{F}$ $\quad k^{-1}_{comp}$ *

where $p = \sqrt{\frac{|S|}{EI_z}}$

axial force → need this!

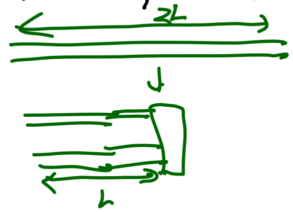



① Substrate contracts faster than film → looks like film wants to expand against the substrate

*– Find S:

① If polySi strain is $\epsilon_r$, then shuttle effectively expands by $\Delta L_s = \epsilon_r L_s$

② This then applies a load to the beams, $\Delta L = \Delta L_s$




③ Beam Stress:

$$\epsilon_b = \frac{\Delta L}{2L} = \frac{\Delta L_s}{2L} = \pm \epsilon_r \frac{L_s}{2L}$$

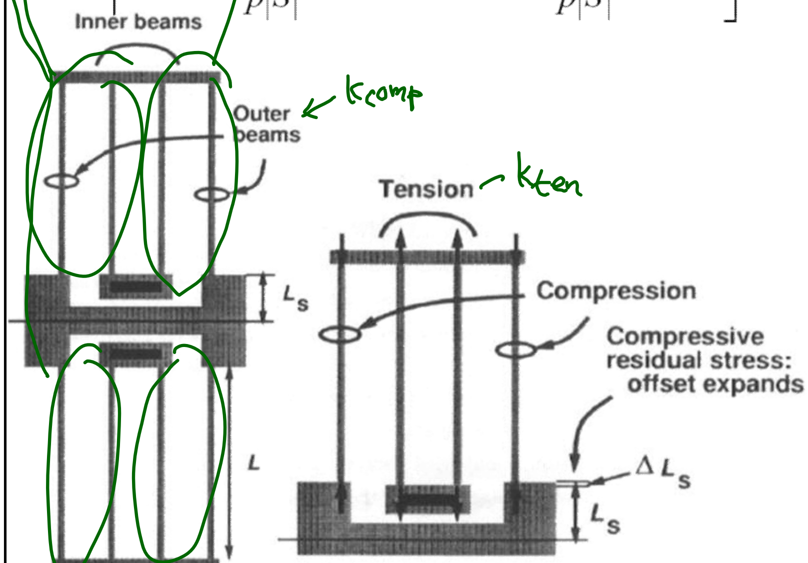
Stress Force:

$$S = \pm E\epsilon_r \left(\frac{L_s}{2L}\right) Wh$$ (axial tension)

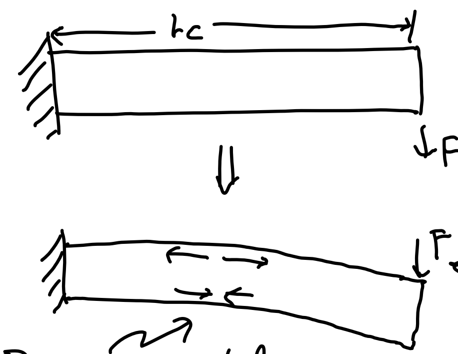
④ Spring Constants:

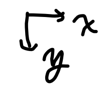
$$k = 4\left(k_{comp}^{-1} + k_{ten}^{-1}\right)^{-1}$$

$$k = 4\left[\frac{-pL + 2\tan(pL/2)}{p|S|} + \frac{pL - 2\tanh(pL/2)}{p|S|}\right]^{-1}$$




Same problem as before: Take a beam & apply a force.




① Apply force.

② Beam responds by bending.

③ This force does work:
$W = F \cdot y(L_c)$

④ Strain generated
↳ so the beam has received an influx of stored energy
↳ magnitude of " " determined by shape

⑤ Then

$U$ = Stored Energy − Work Done ⟶ 0

when we choose the right shape
↓
the right $y = f(x, F)$
↓
then yields k (stiffness)

## Fundamentals: Energy Density

### General Definition of Work:

$$\mathcal{W}(q_1) = \int_0^{q_1} e(q)\, dq$$

q: displacement
e: effort

for EE: $\mathcal{W}(Q) = \int_0^Q \frac{Q}{C} dQ$

### Strain Energy Density

$$w = \int_0^{\epsilon_x} \sigma_x d\epsilon_x$$

$\epsilon_x$ ← Value of strain @ position $(x, y, z)$

$\sigma_x(\epsilon_x) \rightarrow$ relates stress to strain @ position $(x, y, z)$

$[\sigma_x = E\epsilon_x]$

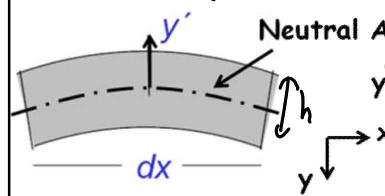
$$w = \int_0^{\epsilon_x} E\epsilon_x d\epsilon_x = \frac{1}{2}E\epsilon_x^2$$

### Total Strain Energy: [J]

$$\mathcal{W} = \iiint \left(\frac{1}{2}E\left(\epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2\right) + \frac{1}{2}G\left(\gamma_{xy}^2 + \gamma_{xz}^2 + \gamma_{yz}^2\right)\right) \underbrace{dV}_{\text{volume}}$$

## Bending Energy Density

W = width into the plane




same as z before

y(x) = transverse displacement of neutral axis

First, find the bending energy $d\mathcal{W}_{bend}$ in an infinitesimal length $dx$

$$d\mathcal{W}_{bend} = W dx \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{2}E\epsilon_x^2(y')\, dy'$$

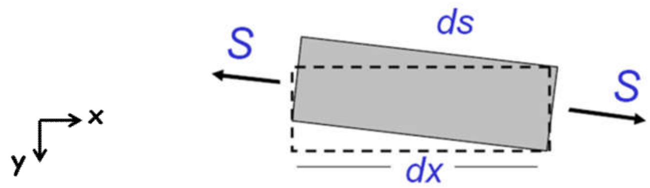
$$\left[\frac{1}{R} = \frac{d^2y}{dx^2},\ \epsilon_x = \frac{y'}{R}\right] \Rightarrow \epsilon_x(y') = y'\frac{d^2y}{dx^2}$$

$$d\mathcal{W}_{bend} = W dx \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{2}E\left[y'\frac{d^2y}{dx^2}\right]^2 dy'$$

$$= \frac{1}{2}E\underbrace{\left(\frac{Wh^3}{12}\right)}_{I_z}\left(\frac{d^2y}{dx^2}\right)^2 dx$$

$$\therefore \boxed{\mathcal{W}_{bend} = \frac{1}{2}EI_z\int_0^L\left(\frac{d^2y}{dx^2}\right)^2 dx}$$

## Energy Due to Axial Load




⇒ energy related lengthening:

$$ds = \left[(dx)^2 + (dy)^2\right]^{1/2} = dx\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2}$$

binomial theorem

$$\simeq dx\left[1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2\right]$$

$$\therefore\ \epsilon_x = \frac{ds - dx}{dx} = \frac{1}{2}\left(\frac{dy}{dx}\right)^2$$

$$\left[dW_{axial} = S\epsilon_x dx = \frac{1}{2} S\left(\frac{dy}{dx}\right)^2 dx\right]$$

$$\boxed{W_{axial} = \frac{1}{2} S\int_0^L \left(\frac{dy}{dx}\right)^2 dx}$$

Axial Strain Energy

⇒ Look @ shear strain energy in your module.

- Go through Module 9, slides 10-18