Lecture 17: Resonance Frequency

- **Announcements:**
- HW#5 online since last Thursday, due next Tuesday, 10 a.m.
- Module 10 on "Resonance Frequency" online
- Midterm Exam next week, Thursday, March 22, 11-12:30, 3109 Etcheverry (right here)
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- Reading: Senturia, Chpt. 10: §10.5, Chpt. 19
- Lecture Topics:
  - Estimating Resonance Frequency
  - Lumped Mass-Spring Approximation
  - ADXL-50 Resonance Frequency
  - Distributed Mass & Stiffness
  - Folded-Beam Resonator
  - Resonance Frequency Via Differential Equations
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- **Last Time:**
- Finished Energy Methods
- Now, start Resonance Frequency via Module 10

Estimating Resonance Frequency

$\mapsto x(t) = X_o \cos\omega_o t$

Potential Energy

$$W(t) = \frac{1}{2}kx^2(t) = \frac{1}{2}kX_o^2\cos^2\omega_o t$$

Kinetic Energy

$$K(t) = \frac{1}{2}m\dot{x}^2(t) = \frac{1}{2}mX_o^2\omega_o^2\sin^2\omega_o t$$

$\dot{x} = \frac{dx}{dt} =$ velocity

Remarks.

(1) Energy must be conserved.

(2) Total Energy = Potential Energy + Kinetic Energy at all times & locations on the structure

$$W_{max} = \frac{1}{2}kX_o^2 = K_{max} = \frac{1}{2}m\omega_o^2X_o^2$$

maximum potential energy

peak displacement

maximum kinetic energy

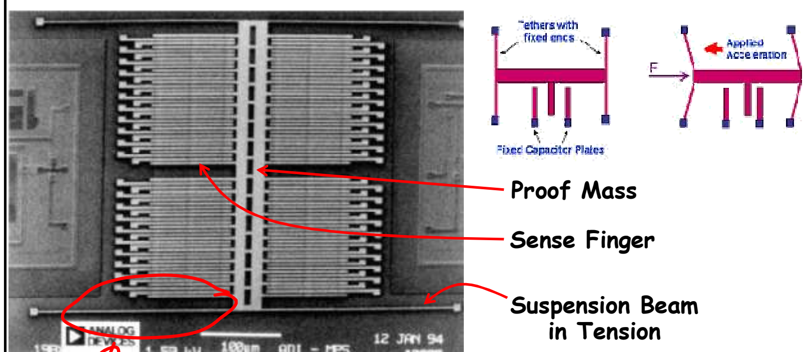
radian frequency

*

★ → $\boxed{\omega_0 = \sqrt{\frac{k}{m}}}$ ⇒ good for problems where mass & stiffness can be separated

i.e., they are distinct

- The proof mass of the ADXL-50 is many times larger than the effective mass of its suspension beams
  - ↳ Can ignore the mass of the suspension beams (which greatly simplifies the analysis)
- Suspension Beam: L = 260 μm, h = 2.3 μm, W = 2 μm

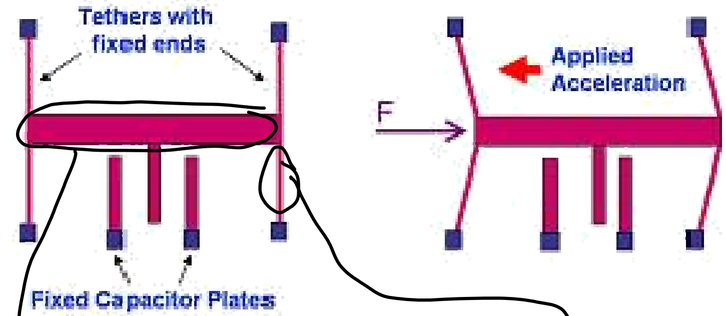


In fabrication: purposely introduce a tensile stress in the beams!

a large one → why?

to avoid compression at all cost

buckling → dead device



mass of structure >> mass of the springs

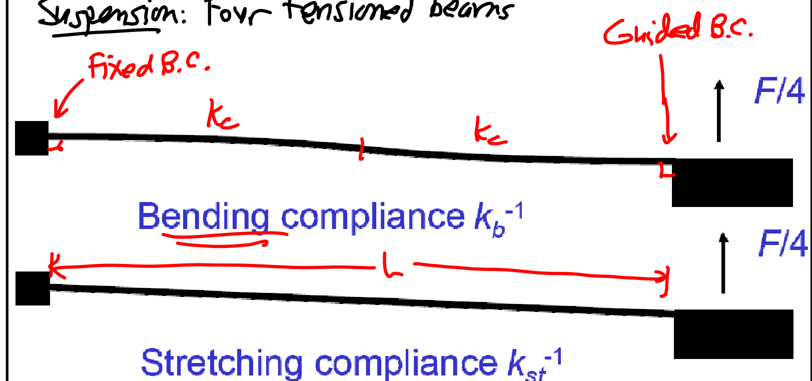
∴ ignore the mass of the springs

stiffness of the springs << stiffness of structure

∴ ignore the stiffness of the structure

For the ADXL-50, 60% the mass comes from the sense fingers → M = 162 ng

Suspension: Four tensioned beams

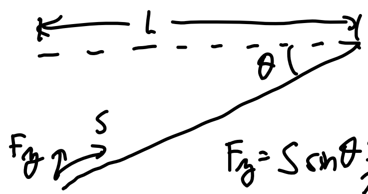


Bending Contribution

$$k_b = k_c \| k_c = \left(\frac{1}{k_c}+\frac{1}{k_c}\right)^{-1} = \frac{k_c}{2} = \frac{1}{2}\,\frac{3E(wh^3/12)}{(L/2)^3}$$

$$\Rightarrow k_b = Ew\left(\frac{h}{L}\right)^3 = 0.24\ \text{N/m}$$

Stretching Contribution



$$F_y = S\sin\theta \simeq S\left(\frac{y}{L}\right) = \underbrace{\left(\frac{S}{L}\right)}_{k_{st}}y$$

(assume small displacement)

k_{st} ← stretching stiffness

$$k_{st} = \frac{S}{L} = \frac{\sigma_r wh}{L} = 0.88\ \text{N/m}$$

Get the total spring constant

bending stiffness
stretching stiffness } parallel → add!

$$k_{tot} = 4(k_b + k_{st}) = 4(0.24 + 0.88) = 4.5\ \text{N/m}$$

Now, get the resonance freq:

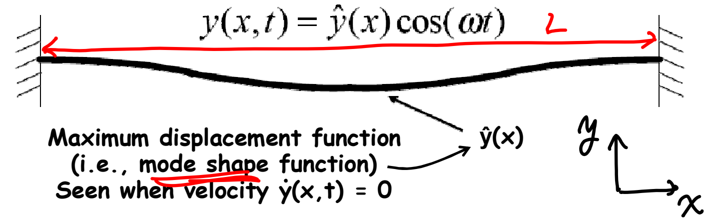
$$f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{4.48\ \text{N/m}}{162\times10^{-12}\ \text{kg}}} = \boxed{26.5\ \text{kHz}}$$

ADXL-50 Data Sheet: $f_o = 24$ kHz ↔ difference?

→ capacitive transducer → electrical stiffness

Find the Resonance Frequency When Mass & Stiffness Are Distributed

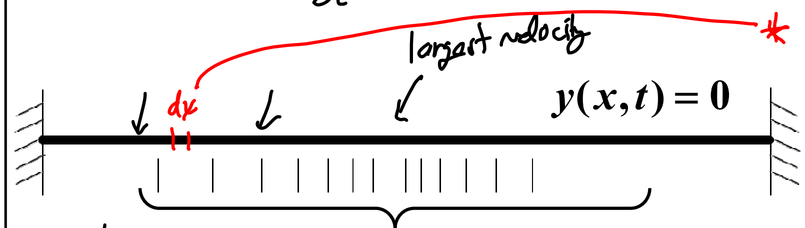
- Vibrating structure displacement function:



- Procedure for determining resonance frequency:
  - Use the static displacement of the structure as a trial function and find the strain energy W_{max} at the point of maximum displacement (e.g., when t=0, π/ω, …)
  - Determine the maximum kinetic energy when the beam is at zero displacement (e.g., when it experiences its maximum velocity)
  - Equate energies and solve for frequency

Get Maximum Kinetic Energy

Velocity: $v(x,t) = \frac{\partial y(x,t)}{\partial t} = -\omega\hat{y}(x)\sin\omega t$



when $y(x,t)=0$, all the energy in the structure is kinetic ($W=0$) ← $t = \frac{\pi}{2\omega}, \frac{3\pi}{2\omega}, \ldots$

$$v\left(x, \frac{(2m+1)\pi}{2\omega}\right) = -\omega\hat{y}(x)$$

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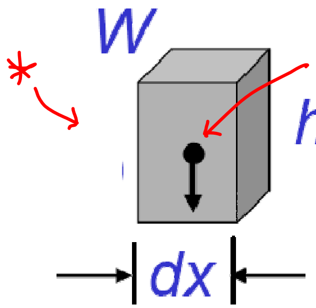


velocity: $v = -\omega \hat{y}(x)$ $\quad \frac{(2m+1)\pi}{2W}$

$dK = \frac{1}{2} \cdot dm \cdot [v(x,t)]^2$

$dm = \rho (W h\, dx)$

↑ density

Maximum Kinetic Energy, K_{max}:

$$K_{max} = \int_0^L \frac{1}{2} \rho W h\, dx\, v^2(x,t) = \int_0^L \frac{1}{2} \rho W h \omega^2 [\hat{y}(x)]^2 dx$$

To get frequency: $K_{max} = W_{max}$

$$\therefore \quad \omega = \sqrt{\frac{W_{max}}{\int_0^L \frac{1}{2} \rho W h [\hat{y}(x)]^2 dx}} \quad [\text{radians/s}]$$

ω = radian resonance freq.

W_{max} = maximum potential energy

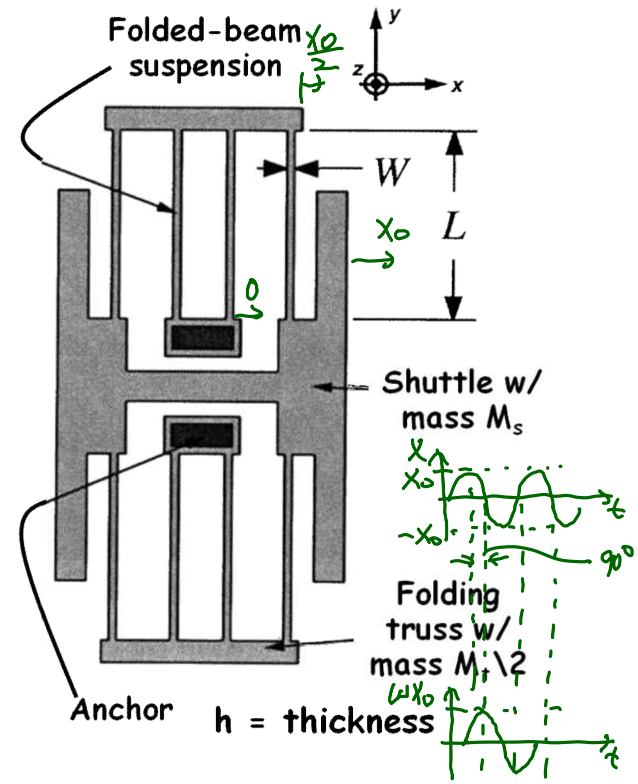
ρ = density of the structural material

W = beam width

h = " thickness

$\hat{y}(x)$ = resonance mode shape

Resonance Freq. of a Folded Beam Structure



- Derive an expression for the resonance frequency of the above structure

Approximation:

⇒ m = shuttle mass

⇒ $k = k_c$

$\Big\} \quad \omega_o = \sqrt{\frac{k_c}{m}}$

↑ But not accurate enough for some applications.

⇒ for better accuracy, must integrate

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Use the Rayleigh-Ritz Method: (energy method)

$\mathcal{K}_{max} = W_{max} \leftarrow \frac{1}{2}kx^2$

Find the kinetic energy → one piece at a time

$$\mathcal{K}_{max} = \underbrace{\mathcal{K}_s}_{\text{shuttle}} + \underbrace{\mathcal{K}_t}_{\text{truss}} + \underbrace{\mathcal{K}_b}_{\text{beams}}$$

$$= \frac{1}{2}v_s^2 M_s + \frac{1}{2}v_t^2 M_t + \frac{1}{2}\int v_b^2 \, dM_b$$

Velocity of the Shuttle: $v_s = \omega_0 X_0$ ← res. freq. ; ← maximum displacement of shuttle

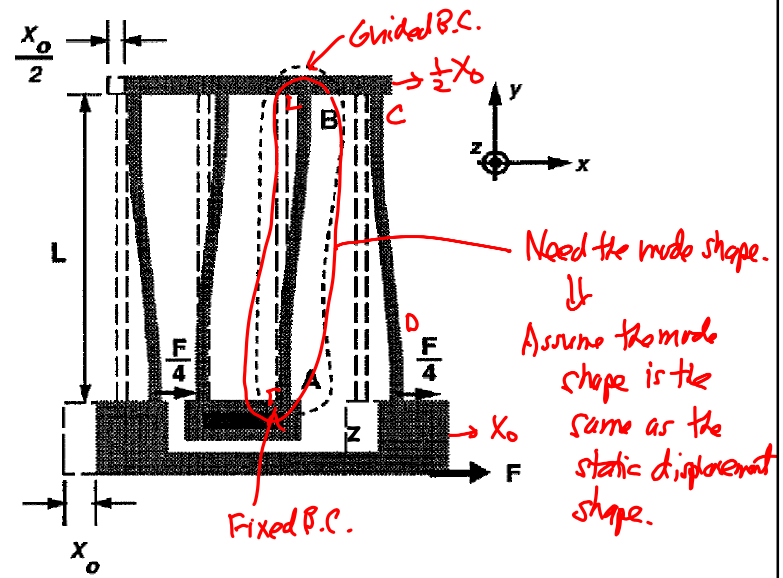
$$\therefore \mathcal{K}_s = \frac{1}{2}v_s^2 M_s = \boxed{\frac{1}{2}\omega_0^2 X_0^2 M_s = \mathcal{K}_s}$$

Velocity of Truss: $v_t = \frac{1}{2}v_s = \frac{1}{2}\omega_0 X_0$

$$\therefore \mathcal{K}_t = \frac{1}{2}\left(\frac{1}{2}\omega_0 X_0\right)^2 M_t = \boxed{\frac{1}{8}\omega_0^2 X_0^2 M_t = \mathcal{K}_t}$$

mass of both trusses

Velocity of the Beam Segments: → first beam [AB]



Segment [AB]:

$$\hat{x}(y) = \frac{F_x}{48EI_z}\left(3Ly^2 - 2y^3\right), \quad 0 \le y \le L \qquad (1)$$

At $y = L$: $x(L) = \frac{X_0}{2} = \frac{F_x L^3}{48EI_z}$ ← B.C.

Substitute into (1):

$$\hat{x}(y) = \frac{X_0}{2}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]$$

which yields for velocity:

$$v_b(y)\Big|_{[AB]} = \frac{X_0}{2}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]\omega_0$$

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Plugging into expression for $\mathcal{K}_b$:

$$\mathcal{K}_{[AB]} = \frac{1}{2}\int_0^L \frac{X_0^2\omega_0^2}{4}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]^2 dM_{[AB]}$$

$$= \frac{X_0^2\omega_0^2}{8}\underbrace{\frac{M_{[AB]}}{L}}_{\text{mass per unit length}}\int_0^L \left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]^2 dy$$

$M_{[AB]}$ = static mass

$$\boxed{\mathcal{K}_{[AB]} = \frac{13}{280} X_0^2 \omega_0^2 M_{[AB]}}$$

For segment [CD]:

$$v_b(y)\Big|_{[CD]} = X_0\left[1 - \frac{3}{2}\left(\frac{y}{L}\right)^2 + \left(\frac{y}{L}\right)^3\right]\omega_0$$

Thus:

$$\mathcal{K}_{[CD]} = \frac{X_0^2\omega_0^2 M_{[CD]}}{2L}\int_0^L \left[1 - \frac{3}{2}\left(\frac{y}{L}\right)^2 + \left(\frac{y}{L}\right)^3\right]^2 dy$$

$$\boxed{\mathcal{K}_{[CD]} = \frac{83}{280} X_0^2 \omega_0^2 M_{[CD]}}$$

static mass of beam [CD]

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