Lecture 18: Equivalent Circuits I

- Announcements:
- Module 11 on Equivalent Circuits online
- HW#5 due Tuesday, 10 a.m., after which you will get solutions
- Midterm Exam next week, Thursday, March 22, 11-12:30, 3109 Etcheverry (right here)
- Passed out solutions to one more old midterm
- Last midterm solutions will be available on Monday in the box outside my office after 10 a.m.
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- Reading: Senturia, Chpt. 10: §10.5, Chpt. 19
- Lecture Topics:
  - Estimating Resonance Frequency
  - Lumped Mass-Spring Approximation
  - ADXL-50 Resonance Frequency
  - Distributed Mass & Stiffness
  - Folded-Beam Resonator
  - Resonance Frequency Via Differential Equations
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- Reading: Senturia, Chpt. 5
- Lecture Topics:
  - Lumped Mechanical Equivalent Circuits
  - Electromechanical Analogies
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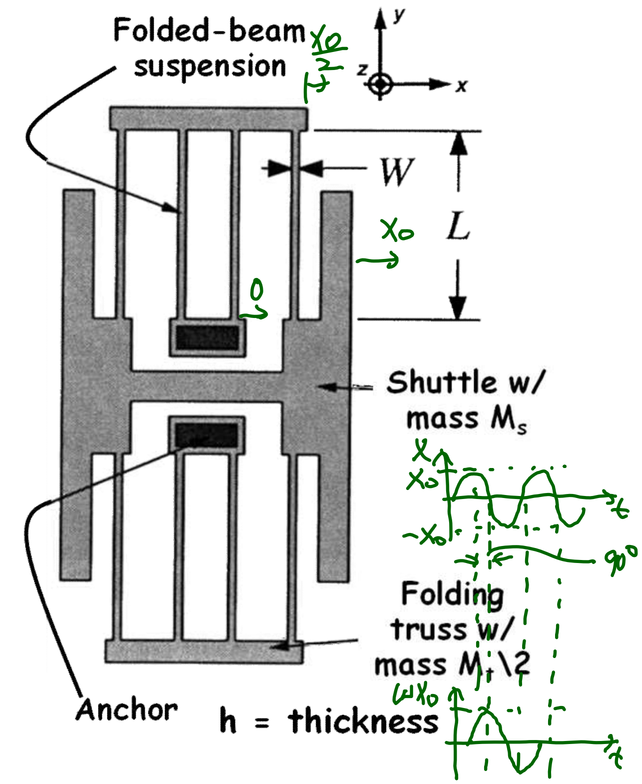
- Last Time:
- Determining resonance frequency for a folded-beam suspended device
- Finish this, then go through Module 10, slides 21-31

Resonance Freq. of a Folded Beam Structure



- Derive an expression for the resonance frequency of the above structure

Approximation)

$\Rightarrow m =$ shuttle mass
$\Rightarrow k = k_c$

$\omega_o = \sqrt{\frac{k_c}{m}}$

But not accurate enough for some applications.

$\Rightarrow$ for better accuracy, must integrate

Use the Rayleigh-Ritz Method: (energy method)

$$\mathcal{K}_{max} = \mathcal{W}_{max} \leftarrow \frac{1}{2}kx^2$$

Find the kinetic energy → one piece at a time

$$\mathcal{K}_{max} = \underbrace{\mathcal{K}_s}_{\text{shuttle}} + \underbrace{\mathcal{K}_t}_{\text{truss}} + \underbrace{\mathcal{K}_b}_{\text{beams}}$$

$$= \frac{1}{2}v_s^2 M_s + \frac{1}{2}v_t^2 M_t + \frac{1}{2}\int v_b^2\, dM_b$$

Velocity of the Shuttle: $v_s = \omega_o X_o$

res. freq. (ω_o); maximum displacement of shuttle (X_o)

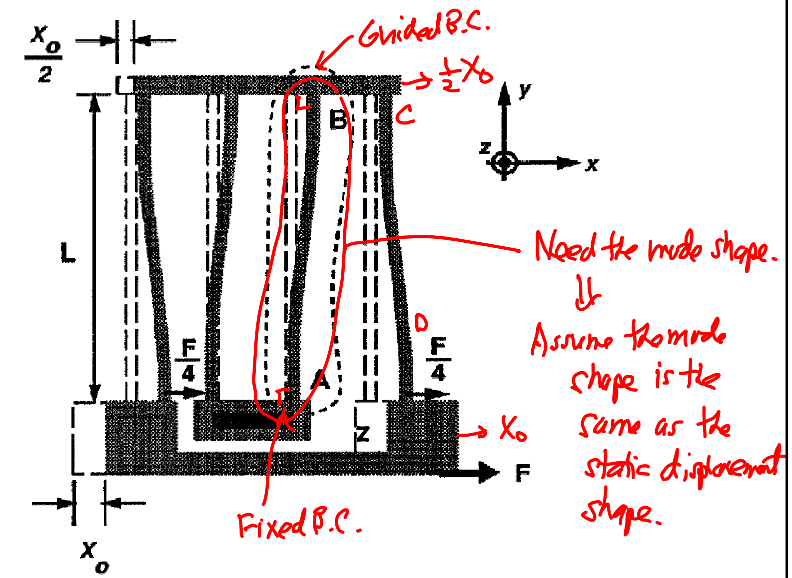
$$\therefore \mathcal{K}_s = \frac{1}{2}v_s^2 M_s = \boxed{\frac{1}{2}\omega_o^2 X_o^2 M_s = \mathcal{K}_s}$$

Velocity of Truss: $v_t = \frac{1}{2}v_s = \frac{1}{2}\omega_o X_o$

$$\therefore \mathcal{K}_t = \frac{1}{2}\left(\frac{1}{2}\omega_o X_o\right)^2 M_t = \boxed{\frac{1}{8}\omega_o^2 X_o^2 M_t = \mathcal{K}_t}$$

mass of both trusses (M_t)

Velocity of the Beam Segments: → first beam [AB]



Segment [AB]:

$$\hat{x}(y) = \frac{F_x}{48EI_z}\left(3Ly^2 - 2y^3\right), \quad 0 \le y \le L \qquad (1)$$

At $y = L$: $x(L) = \frac{X_o}{2} = \frac{F_x L^3}{48EI_z}$ ← B.C.

Substitute into (1):

$$\hat{x}(y) = \frac{X_o}{2}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]$$

which yields for velocity:

$$v_b(y)\Big|_{[AB]} = \frac{X_o}{2}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]\omega_o$$

Plugging into expression for $\mathcal{K}_b$:

$$\mathcal{K}_{[AB]} = \frac{1}{2}\int_0^L \frac{X_o^2\omega_o^2}{4}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]^2 dM_{[AB]}$$

$$= \frac{X_o^2\omega_o^2 M_{[AB]}}{8L}\int_0^L \left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]^2 dy$$

mass per unit length

$M_{[AB]}$ = static mass

$$\boxed{\mathcal{K}_{[AB]} = \frac{13}{280}X_o^2\omega_o^2 M_{[AB]}}$$

For segment [CD]:

$$v_b(y)\big|_{[CD]} = X_o\left[1 - \frac{3}{2}\left(\frac{y}{L}\right)^2 + \left(\frac{y}{L}\right)^3\right]\omega_o$$

Thus:

$$\mathcal{K}_{[CD]} = \frac{X_o^2\omega_o^2 M_{[CD]}}{2L}\int_0^L \left[1 - \frac{3}{2}\left(\frac{y}{L}\right)^2 + \left(\frac{y}{L}\right)^3\right]^2 dy$$

$$\boxed{\mathcal{K}_{[CD]} = \frac{83}{280}X_o^2\omega_o^2 M_{[CD]}}$$

static mass of beam [CD]

Let $M_b \triangleq$ total mass of all 8 beams

Thus:

$$\mathcal{K}_b = 4\mathcal{K}_{[AB]} + 4\mathcal{K}_{[CD]} = \frac{6}{35}X_o^2\omega_o^2 M_b$$

and

$$\boxed{\mathcal{K}_{max} = X_o^2\omega_o^2\left[\frac{1}{2}M_s + \frac{1}{8}M_t + \frac{6}{35}M_b\right]}$$

both trusses — all beams

for the total mechanical ckt.

$W_{max} \rightarrow$ max. potential energy $\rightarrow$ equal to the work done to achieve maximum deflection

$$W_{max} = \frac{1}{2}k_x X_o^2$$

Then, using Rayleigh-Ritz:

$$\mathcal{K}_{max} = W_{max}$$

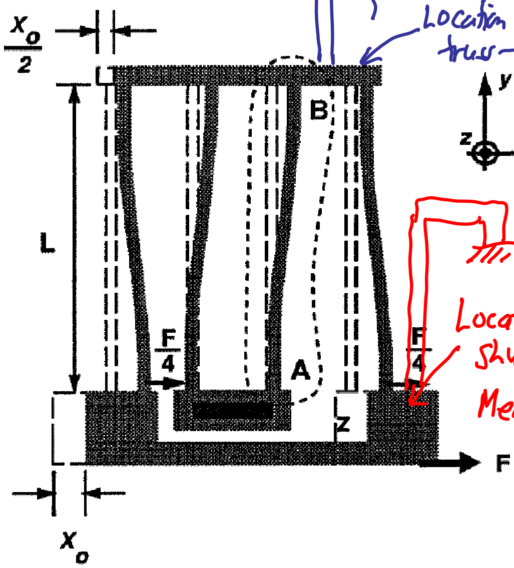
$$\cancel{X_o^2}\,\omega_o^2\left[\frac{1}{2}M_s + \frac{1}{8}M_t + \frac{6}{35}M_b\right] = \frac{1}{2}\cancel{k_x}\cancel{X_o^2}$$

k_c

$$\boxed{\omega_o = \left[\frac{k_c}{M_{eq}}\right]^{\frac{1}{2}} \quad \text{where } M_{eq} = M_s + \frac{1}{4}M_t + \frac{12}{35}M_b}$$

(Resonance Freq. of a Folded-Beam Suspended Shuttle)

Equivalent Dynamic Mass



← spring
mass
Location on Folding truss → $M_{eq(truss)}$ *

← spring
Location on shuttle: $M_{eq(shuttle)}$

Equivalent Mass

$$\text{Equiv. Mass} = M_{eq,x} = \frac{K_{max}}{\frac{1}{2}v_x^2} = \frac{\frac{1}{2}\rho A \int_0^{\ell} v^2(x)\,dx}{\frac{1}{2}v_x^2}$$

↑ velocity at location x

$$M_{eq(shuttle)} = \frac{K_{max}}{\frac{1}{2}v_{shuttle}^2} = \frac{\cancel{\omega_o^2 x_o^2 \left(\tfrac{1}{2}\right)}\left[M_s + \frac{1}{4}M_t + \frac{12}{35}M_b\right]}{\cancel{\frac{1}{2}\omega_o^2 x_o^2}}$$

$$M_{eq(shuttle)} = M_s + \frac{1}{4}M_t + \frac{12}{35}M_b$$

static masses

*

$$M_{eq(truss)} = \frac{\cancel{\omega_o^2 x_o^2 \left(\tfrac{1}{2}\right)}\left[M_s + \frac{1}{4}M_t + \frac{12}{35}M_b\right]}{\cancel{\tfrac{1}{2}}\left(\frac{1}{4}\right)\cancel{\omega_o^2 x_o^2}}$$

$$M_{eq(truss)} = 4\left[M_s + \frac{1}{4}M_t + \frac{12}{35}M_b\right] = 4M_{eq(shuttle)}$$

Equiv. Dynamic Mass

Equiv. Dynamic Stiffness

$$\omega_o = \sqrt{\frac{k_{eq}(x)}{M_{eq}(x)}} \rightarrow \boxed{k_{eq}(x) = \omega_o^2 M_{eq}(x)}$$

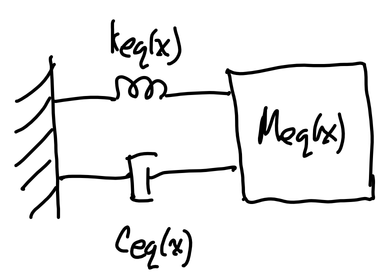
Equiv. Dynamic Damping

$$Q = \frac{\omega_o M_{eq}(x)}{c_{eq}(x)} \rightarrow c_{eq}(x) = \frac{\omega_o M_{eq}(x)}{Q} = \frac{\sqrt{k_{eq}(x) M_{eq}(x)}}{Q}$$

$\omega_o M_{eq}(x) \rightarrow L$

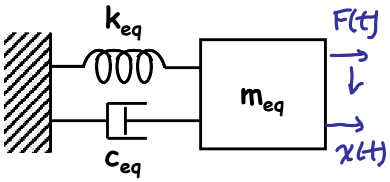
$c_{eq}(x) \rightarrow R$

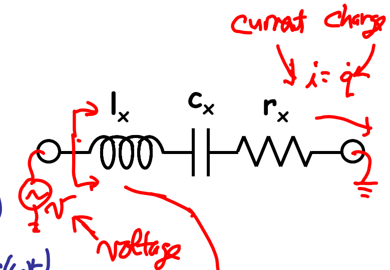
↑ damping



specified @ a single location x

Electromechanical Analogies





$F(t) = F\cos(\omega t) \rightarrow x(t) = X\cos(\omega t)$
(off resonance)

Equation of Motion:

$$m_{eq}\ddot{x} + c_{eq}\dot{x} + k_{eq}x = F(t)$$

$\Rightarrow$ using phasor concepts:

$$F = j\omega m_{eq}\dot{x} + \frac{k_{eq}}{j\omega}\dot{x} + c_{eq}\dot{x}$$

Impedance Looking in:

$$\frac{v}{i} = j\omega l_x + \frac{1}{j\omega c_x} + r_x$$

$$v = j\omega l_x i + \frac{(1/c_x)}{j\omega} i + r_x i$$

Compare

$\Rightarrow$ by analogy

$F \rightarrow v$	$m_{eq} \rightarrow l_x$	
$\dot{x} \rightarrow i$	$k_{eq} \rightarrow \frac{1}{c_x}$	$c_{eq} \rightarrow r_x$

[Parameter Relationships in the Current Analogy]

