Lecture 18: Equivalent Circuits I

- Announcements:
  - Module 11 on Equivalent Circuits online
  - HW#5 due Tuesday, 10 a.m., after which you will get solutions
  - Midterm Exam next week, Thursday, March 22, 11-12:30, 3109 Etcheverry (right here)
  - Passed out solutions to one more old midterm
  - Last midterm solutions will be available on Monday in the box outside my office after 10 a.m.

- Reading: Senturia, Chpt. 10: §10.5, Chpt. 19
- Lecture Topics:
  - Estimating Resonance Frequency
  - Lumped Mass-Spring Approximation
  - ADXL-50 Resonance Frequency
  - Distributed Mass & Stiffness
  - Folded-Beam Resonator
  - Resonance Frequency Via Differential Equations

- Reading: Senturia, Chpt. 5
- Lecture Topics:
  - Lumped Mechanical Equivalent Circuits
  - Electromechanical Analogies

- Last Time:
- Determining resonance frequency for a folded-beam suspended device
- Finish this, then go through Module 10, slides 21-31

Derive an expression for the resonance frequency of the above structure:

\[
\omega_0 = \sqrt{\frac{k}{m}}
\]

But not accurate enough in some applications.

\[\Rightarrow\] for better accuracy, must integrate
Use the Rayleigh–Ritz Method: (energy method)

\[ K_{\text{max}} = \frac{1}{2} k x^2 \]

Find the kinetic energy at one piece at a time

\[ K_{\text{max}} = K_s + K_t + K_b \]

- shuttle
- trans beams

\[ = \frac{1}{2} M_s\dot{x}_s^2 + \frac{1}{2} M_t\dot{x}_t^2 + \frac{1}{2} \int M_b \ddot{y}^2 \, dy \]

Velocity of the shuttle: \( v_s = \omega_0 x_0 \)

Velocity of trans: \( v_t = \frac{1}{2} v_s \)

\[ K_s = \frac{1}{2} M_s\dot{x}_s^2 = \frac{1}{2} M_t\dot{x}_t^2 = \frac{1}{2} \int M_b \ddot{y}^2 \, dy \]

Velocity of the beam segments

\[ \ddot{x}(y) = \frac{F_x}{4EI_z} \left( 3y^2 - 2y^3 \right), 0 \leq y \leq L \]

At \( y = 0 \):

\[ \dot{x}(0) = \frac{F_x}{2} \frac{L^3}{4EI_z} \]

\[ \text{Fixed B.C.} \]

Substitute into (1):

\[ \ddot{x}(y) = \frac{F_x}{2} \left( 3y^2 - 2y^3 \right) \]

Which yields the velocity:

\[ \dot{x}(y) \bigg|_{AB} = \frac{F_x}{2} \left[ 3 \left( \frac{y}{L} \right)^2 - 2 \left( \frac{y}{L} \right)^3 \right] \]

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Plugging into expression for $K_b$:

$$K_{[AB]} = \frac{1}{2} \int_0^L \frac{x^2 w_0^2}{4} \left[3 \left( \frac{x}{L} \right)^2 - 2 \left( \frac{x}{L} \right)^3 \right]^2 dM_{[AB]}$$

$$= \frac{x_o \omega_0^2 M_{[AB]}}{8L} \int_0^L \left[3 \left( \frac{y}{L} \right)^2 - 2 \left( \frac{y}{L} \right)^3 \right]^2 dy$$

$K_{[AB]} = \text{static mass}$

$$K_{[AB]} = \frac{13}{280} x_o \omega_0^2 M_{[AB]}$$

For segment [CD]:

$$N_b(y) \bigg|_{[CD]} = x_o \left[1 - \frac{3}{2} \left( \frac{y}{L} \right)^2 + \left( \frac{y}{L} \right)^3 \right] \omega_0$$

Thus:

$$K_{[CD]} = \frac{x_o \omega_0^2 M_{[CD]}}{2L} \int_0^L \left[1 - \frac{3}{2} \left( \frac{y}{L} \right)^2 + \left( \frac{y}{L} \right)^3 \right]^2 dy$$

$$K_{[CD]} = \frac{\frac{2}{280} x_o \omega_0^2 M_{[CD]}}{8}$$

Let $M_b = \text{total mass of all}$

$\text{beams}$

Thus:

$$K_1 = 4K_{[AB]} + 4K_{[CD]} = \frac{6}{35} x_o \omega_0^2 M_b$$

Then, using Rayleigh–Ritz:

$$K_{\text{max}} = K_{\text{max}}$$

$$x_o^{-2} \omega_0^2 \left[ \frac{1}{2} M_s + \frac{1}{4} M_s + \frac{1}{35} M_b \right] = \frac{1}{x_o^2}$$

$$\omega_0 = \left[ \frac{k_c}{M_{\text{eq}}} \right]^{1/2}$$

where $M_{\text{eq}} = M_s + \frac{1}{4} M_s + \frac{12}{35} M_b$

(Resonance Freq. of a Folded-Beam Suspended Shuttle)
**Equivalent Dynamic Mass**

\[
\text{Equivalent Mass} = M_{eq,x} = \frac{1}{2} V^2_x \frac{x}{x_{ref}}
\]

\[
M_{eq}(\text{shuttle}) = \frac{1}{2} V^2_{\text{shuttle}}
\]

\[
M_{eq}(\text{shuttle}) = M_s + \frac{1}{4} M_b + \frac{12}{35} M_b
\]

\[
\frac{1}{2} \rho A \int_0^L V^2(x) dx
\]

\[
\omega_0^2 = \frac{K_{eq}(x)}{M_{eq}(x)} \rightarrow K_{eq}(x) = \omega_0^2 M_{eq}(x)
\]

**Equation Dynamic Stiffness**

\[
Q = \frac{\omega_0 M_{eq}(x)}{C_{eq}(x)} \rightarrow L_{damping} = \frac{1}{Q}
\]

\[
C_{eq}(x) = \frac{L_{damping} M_{eq}(x)}{Q}
\]

**Specified @ a single location x**
**Electromechanical Analogies**

\[ F(t) = k_{eq}(x(t)) \Rightarrow x(t) = \alpha(t) \cos(\omega t) \] (self resonance)

**Equation of Motion:**

\[ m_{eq}\ddot{x} + c_{eq}\dot{x} + k_{eq}x = F(t) \]

\[ \Rightarrow \text{using phasor concepts:} \]

\[ F = j\omega m_{eq}\omega \dot{x} + \frac{k_{eq}}{j\omega}\dot{x} + c_{eq}\dot{x} \]

**Impedance Looking in:**

\[ \frac{\nu}{i} = j\omega l_{eq} + \frac{1}{j\omega c_{eq}} + r_{x} \]

\[ \nu = j\omega l_{eq} + \frac{1}{j\omega c_{eq}} + r_{x} \]

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\[ \Rightarrow \text{by analogy} \]

\[ F \rightarrow \nu \quad m_{eq} \rightarrow l_{eq} \]

\[ \dot{x} \rightarrow i \quad k_{eq} \rightarrow \frac{1}{c_{eq}} \]

\[ c_{eq} \rightarrow r_{x} \]

[Parameter Relationships in the Current Analogy]