Lecture 19: Capacitive Transducers

- **Announcements:**
- Module 12 on Capacitive Transducers online
- Midterm Exam this coming Thursday, March 22, 11-12:30 p.m., 3109 Etcheverry (right here)
- -----------------------------------
- Reading: Senturia, Chpt. 5
- Lecture Topics:
  - Lumped Mechanical Equivalent Circuits
  - Electromechanical Analogies
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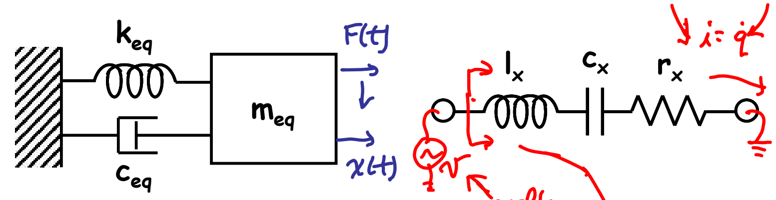
- Reading: Senturia, Chpt. 5, Chpt. 6
- Lecture Topics:
  - Energy Conserving Transducers
    - Charge Control
    - Voltage Control
  - Parallel-Plate Capacitive Transducers
    - Linearizing Capacitive Actuators
    - Electrical Stiffness
  - Electrostatic Comb-Drive
    - 1st Order Analysis
    - 2nd Order Analysis
- -----------------------------------
- **Last Time:**
- Finishing up our first pass on equivalent circuits
- Continue this now …

Electromechanical Analogies



$F(t) = F\cos(\omega t) \rightarrow x(t) = X\cos(\omega t)$
(off resonance)

Equation of Motion:

$$m_{eq}\ddot{x} + c_{eq}\dot{x} + k_{eq}x = F(t)$$

$\Rightarrow$ using phasor concepts:

$$F = j\omega m_{eq}\dot{x} + \frac{k_{eq}}{j\omega}\dot{x} + c_{eq}\dot{x}$$

Impedance Looking in:

$$\frac{v}{i} = j\omega l_x + \frac{1}{j\omega C_x} + r_x$$

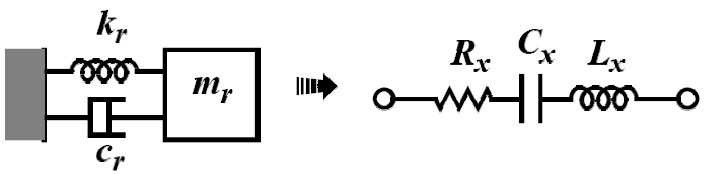
$$v = j\omega l_x i + \frac{(1/C_x)}{j\omega} i + r_x i$$

Compare

$\Rightarrow$ by analogy

$$\boxed{F \rightarrow v \quad m_{eq} \rightarrow l_x \quad \dot{x} \rightarrow i \quad k_{eq} \rightarrow \frac{1}{C_x} \quad c_{eq} \rightarrow r_x}$$

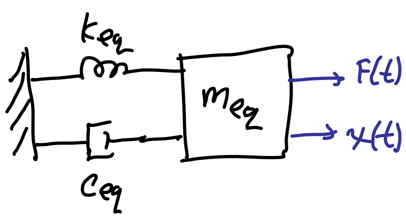
[Parameter Relationships in the Current Analogy]



- Mechanical-to-electrical correspondence in the current analogy:

Mechanical Variable	Electrical Variable
Damping, c	Resistance, R
Stiffness^{-1}, k^{-1}	Capacitance, C
Mass, m	Inductance, L
Force, f	Voltage, V
Velocity, v	Current, I

$\boxed{\text{Lowpass Biquad Transfer Function}}$



$F = j\omega m_{eq}\dot{x} + \frac{k_{eq}}{j\omega}\dot{x} + c_{eq}\dot{x}$ (from last time)

$\Rightarrow$ convert to full phasor form

$$F = (j\omega)(j\omega X)m_{eq} + \frac{k_{eq}}{j\omega}(j\omega X) + c_{eq}(j\omega X)$$

$$\frac{X}{F}(j\omega) = \frac{1}{k_{eq}}\left[-\omega^2\frac{m_{eq}}{k_{eq}} + 1 + j\frac{c_{eq}\omega}{k_{eq}}\right]^{-1}$$

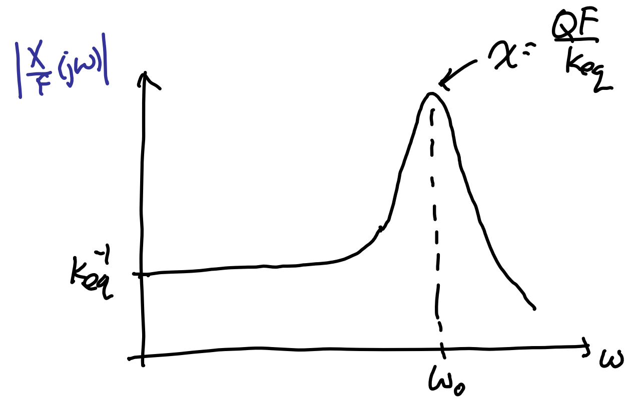
*

$$\left[\frac{k_{eq}}{m_{eq}} = \omega_0^2,\ Q = \frac{m_{eq}\omega_0}{c_{eq}} = \frac{k_{eq}}{\omega_0 c_{eq}} \rightarrow \frac{k_{eq}}{c_{eq}} = Q\omega_0\right]$$

quality factor

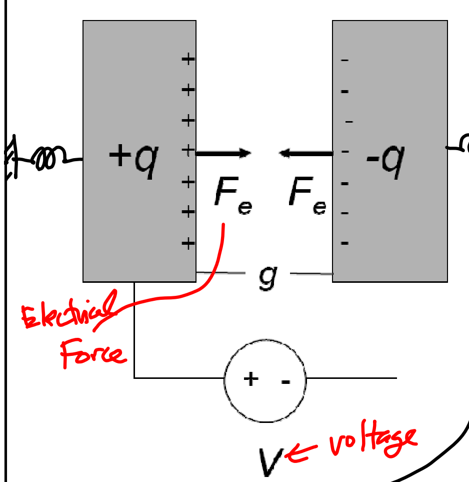
*

$$\frac{X}{F}(j\omega) = \frac{k_{eq}^{-1}}{1 - \left(\frac{\omega}{\omega_0}\right)^2 + j\frac{\omega}{Q\omega_0}}$$



- Go through pages 11-22 of Module 11
- Then, start into Module 12

Basic Physics of Electrostatic Actuation



Assume the plates are supported elastically.

Goal: Determine gap spacing as a function of input variables

1st: Determine the energy of the system.

2nd: Ask: what can I do to change the energy of the system?

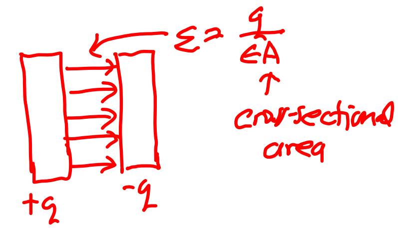
① change the charge q

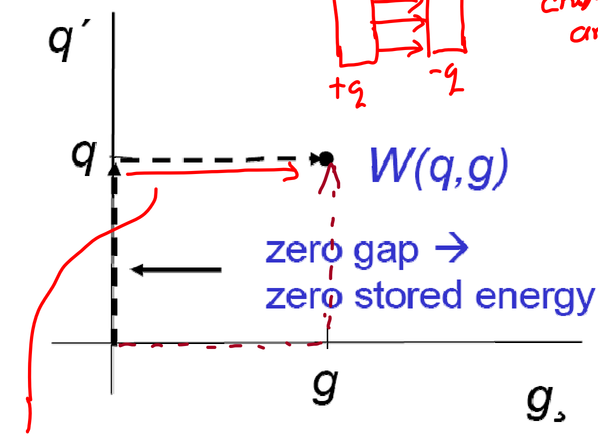
② change the separation g

$$\Delta W(q,g) = V\Delta q + F_e \Delta g$$

$$dW = V\,dq + F_e\,dg$$

Stored Energy





No change in charge: $dq=0$

$$W = 0 + \int_0^g F_e\,dg'$$

$$F_e = \left(\frac{q}{2}\right)\mathcal{E} = \frac{1}{2}\frac{q^2}{\epsilon A} \quad \text{(independent of } g\text{)}$$

$$\therefore W = \int_0^g F_e\,dg' = F_e g'\Big|_0^g = F_e g$$

$$\boxed{W(g) = \frac{1}{2}\frac{q^2}{\epsilon A}\,g}$$

= **

*

* Work done to charge a C to q at fixed gap g.

$$dW = Vdq + F_e \underbrace{dg}_{0}$$

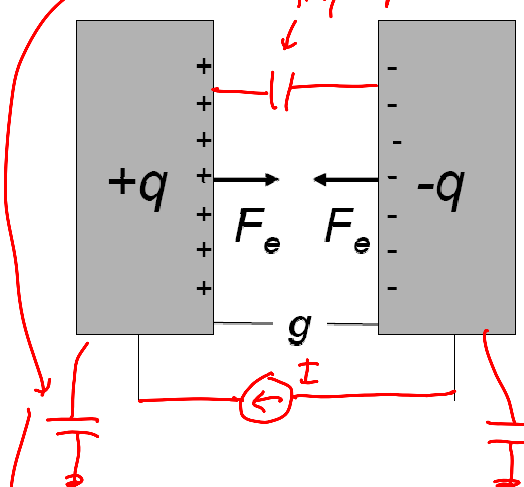
For a capacitor:

$$q = CV \rightarrow V = \frac{q}{C}$$

$$\therefore W(q) = \int_0^q V\,dq = \int_0^q \left(\frac{q}{C}\right) dq = \frac{1}{2}\frac{q^2}{C}$$

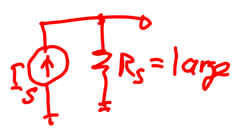
** $$\boxed{W(q) = \frac{1}{2}\frac{q^2}{\epsilon A}\, g}$$

Charge Control Case



Voltage Source:

Current Source:



Thus, I-drive is not practical in many cases.

From $dW = Vdq + F_e dg$

⇒ Force is given by:

$$F_e = \frac{\partial W(q,g)}{\partial g}\bigg|_{q=\text{const.}} = \frac{\partial}{\partial g}\left(\frac{1}{2}\frac{q^2}{\epsilon A}\, g\right)$$

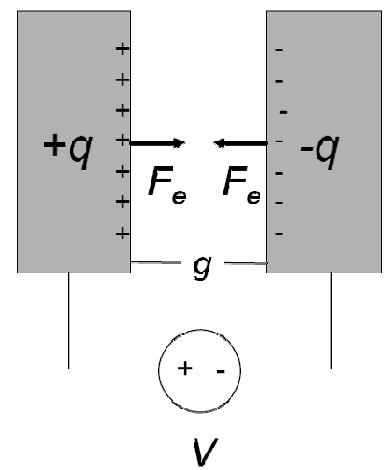
$$\therefore \boxed{F_e = \frac{1}{2}\frac{q^2}{\epsilon A}} \Rightarrow \text{indep. of gap spacing!}$$

⇒ voltage is given by:

$$V = \frac{\partial W(q,g)}{\partial q}\bigg|_{g=\text{const.}} = \frac{\partial}{\partial q}\left(\frac{1}{2}\frac{q^2}{\epsilon A}\, g\right) = \frac{qg}{\epsilon A} \quad \therefore \boxed{V = \frac{q}{C}} \checkmark$$

(consistent w/ what we know)

Voltage Control



Want to write $F_e = f(V)$

We know this:

$$dW = Vdq + F_e dg$$

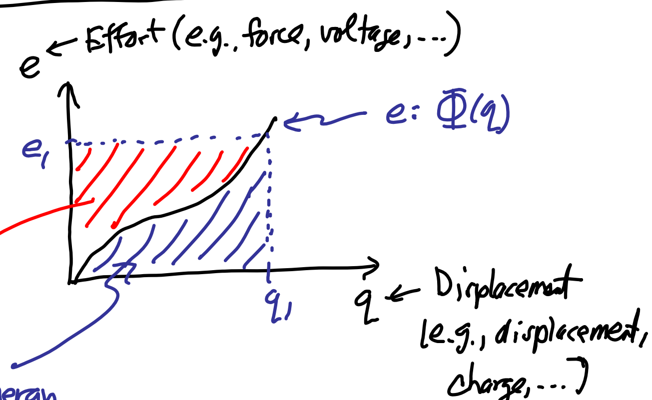
$$W = f(q,g)$$

Need: $\mathcal{W}' = f(V, g)$

want to replace charge q w/ voltage V

Can get this using a legendre transformation.

Energy & Co-Energy



Energy

$$\mathcal{W}(q_1) = \int_0^{q} e\,dq = \int_0^{q_1} \Phi(q)\,dq$$

Co-Energy:

$$\mathcal{W}'(e_1) = \int_0^{e_1} q\,de = \int_0^{e_1} \Phi^{-1}(e)\,de$$

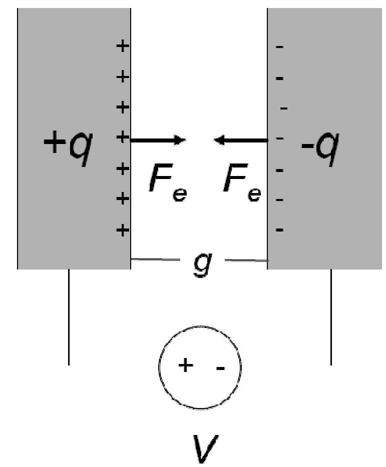
For a linear system, these will be equal.

Can define co-energy as:

$$\mathcal{W}'(e) = eq - \mathcal{W}(q) \quad \text{(from the plot)}$$

energy

Co-Energy Formulation for Voltage-Control



$$* \quad \mathcal{W}'(V, g) = Vq - \mathcal{W}(q, g)$$

Differentially, this becomes

$$d\mathcal{W}'(V, g) = (q\,dV + \cancel{V\,dq}) - d\mathcal{W}(q, g)$$

$$[d\mathcal{W}(q, g) = F_e\,dg + \cancel{V\,dq}]$$

$$\boxed{d\mathcal{W}'(V, g) = q\,dV - F_e\,dg}$$

Working Co-Energy Expression

Find the Co-Energy in terms of voltage, V:

$$\mathcal{W}' = \int_0^V q(g,V')\,dV' = \int_0^V \left(\frac{\epsilon A}{g}\right) V'\,dV'$$

$$= \frac{1}{2}\left(\frac{\epsilon A}{g}\right)V^2 = \frac{1}{2}CV^2 \checkmark$$ (as expected)

Electrostatic (or Voltage-Controlled) Force:

$$F_e = -\left.\frac{\partial \mathcal{W}'(V,g)}{\partial g}\right|_{V=\text{const.}}$$

$$= -\frac{1}{2}\left(-\frac{\epsilon A}{g^2}\right)V^2 = \boxed{\frac{1}{2}\frac{C}{g}V^2 = F_e}$$

depends on gap!

Charge:

$$q = \left.\frac{\partial \mathcal{W}'(V,g)}{\partial V}\right|_{g=\text{const.}} = \frac{\epsilon A}{g}V = CV \checkmark$$

(as expected)

Charge-Control of a Spring-Suspended C