Lecture 20: Pull-In Voltage

- **Announcements:**
- Module 12 on Capacitive Transducers online
- HW#6 online soon (today)
- In mid-class (to make sure everyone is here)
    - Project introduction today
    - Midterm exams coming back today
    - Z-scores presented as well, at the end of class
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- Reading: Senturia, Chpt. 5, Chpt. 6
- Lecture Topics:
    - Energy Conserving Transducers
        - Charge Control
        - Voltage Control
    - Parallel-Plate Capacitive Transducers
        - Linearizing Capacitive Actuators
        - Electrical Stiffness
    - Electrostatic Comb-Drive
        - 1st Order Analysis
        - 2nd Order Analysis
- -----------------------------------
- **Last Time:**
- Introduced co-energy and started analyzing spring-suspended C's

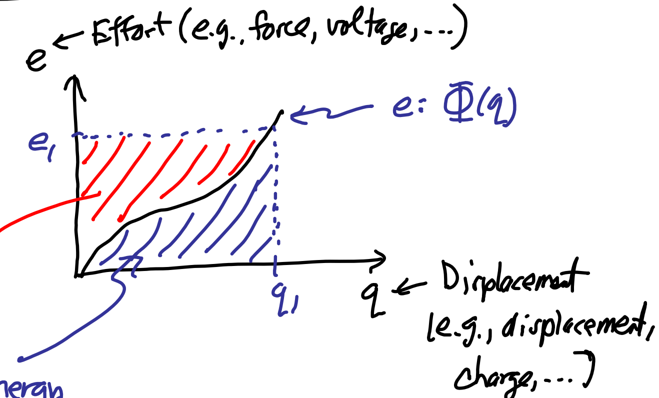
over

Need: $\mathcal{W}' = f(V, g)$

want to replace charge q w/ voltage V

Can get this using a Legendre transformation.

Energy & Co-Energy



Energy

$$\mathcal{W}(q_1) = \int_0^{q_1} e\,dq = \int_0^{q_1} \Phi(q)\,dq$$

Co-Energy:

$$\mathcal{W}'(e_1) = \int_0^{e_1} q\,de = \int_0^{e_1} \Phi^{-1}(e)\,de$$

For a linear system, these will be equal.

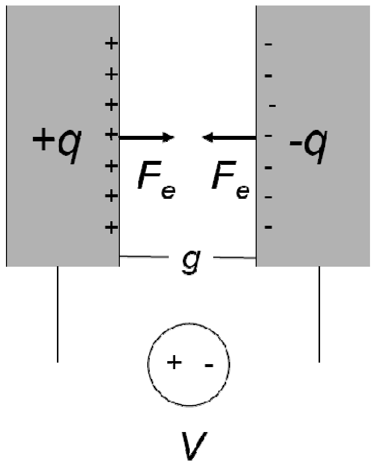
Can define co-energy as:

$$\mathcal{W}'(e) = eq - \mathcal{W}(q) \quad \text{(from the plot)}$$

energy

*

Co-Energy Formulation for Voltage-Control



$* \quad W'(V,g) = Vq - W(q,g)$

Differentially, this becomes

$$dW'(V,g) = (q\,dV + \cancel{V\,dq}) - dW(q,g)$$

$$[dW(q,g) = F_e\,dg + \cancel{V\,dq}]$$

$$\boxed{dW'(V,g) = q\,dV - F_e\,dg}$$

Working Co-Energy Expression

Find the Co-Energy in terms of voltage, V:

$$W' = \int_0^V q(g,V')\,dV' = \int_0^V \left(\frac{\epsilon A}{g}\right) V'\,dV'$$

$$= \frac{1}{2}\left(\frac{\epsilon A}{g}\right)V^2 = \frac{1}{2}CV^2 \checkmark \quad \text{(as expected)}$$

Electrostatic (or Voltage-Controlled) Force:

$$F_e = -\left.\frac{\partial W'(V,g)}{\partial g}\right|_{V=\text{const.}}$$

$$= -\frac{1}{2}\left(-\frac{\epsilon A}{g^2}\right)V^2 = \boxed{\frac{1}{2}\frac{C}{g}V^2 = F_e}$$

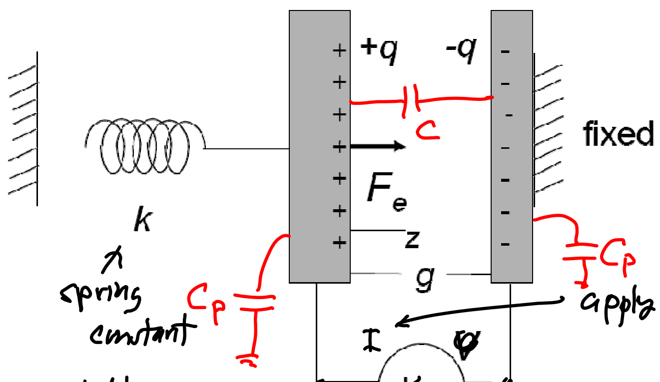
depends on gap!

Charge:

$$q = \left.\frac{\partial W'(V,g)}{\partial V}\right|_{g=\text{const.}} = \frac{\epsilon A}{g}V = CV \checkmark$$

(as expected)

Charge-Control of a Spring-Suspended C



Force generated by charge q (supplied by current I):

$$F_e = \frac{\partial W(q,g)}{\partial g}\Big|_{q=const} = \frac{q^2}{2\epsilon A}$$

Restoring force of spring: $F_{spring} = kz = F_e$ ← equilibrium

The gap:

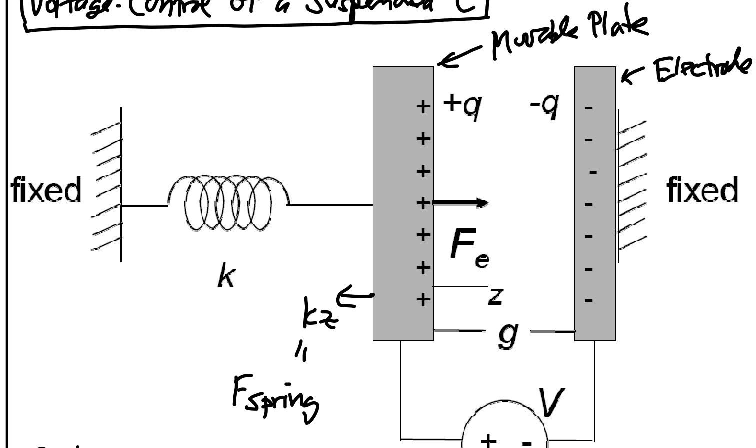
$$g = g_o - z = g_o - \frac{F_e}{k} = \boxed{g_o - \frac{1}{2}\frac{q^2}{\epsilon A}\frac{1}{k} = g}$$

(g_o = initial gap)

↳ $q\uparrow$ can drive $g \to 0$ in a continuous fashion

$$V = \frac{q}{C} = \frac{qg}{\epsilon A} = \boxed{\frac{q}{\epsilon A}\left(g_o - \frac{1}{2}\frac{q^2}{\epsilon A}\frac{1}{k}\right) = V} \rightarrow V\downarrow \text{ as } g\downarrow$$

Voltage-Control of a Suspended C



But now:

$$F_e = \frac{\partial W'(V,g)}{\partial g}\Big|_{q=const.} \Rightarrow F_e = \frac{1}{2}\frac{\epsilon A}{g^2}V^2$$

And the gap:

$$g = g_o - z = g_o - \frac{F_e}{k} = \boxed{g_o - \frac{1}{2}\frac{\epsilon A}{g^2}\frac{V^2}{k} = g}$$

(g_o = initial gap spacing)

g shows up on both sides!

If $V\uparrow \rightarrow g\downarrow \rightarrow F_e\uparrow$ (+) feedback!

If loop gain > 1, then this system will go unstable!

plate will collapse! (into the electrode)

Charge: (for a stable gap)

$$q = \left.\frac{\partial W'(V,g)}{\partial V}\right|_{g=\text{const.}} = CV \checkmark \quad \text{(as expected)}$$

Stability Analysis

⇒ determine under what conditions voltage-control will cause collapse of the plates:

$$F_{net} = F_e - F_{spring} = \underbrace{\frac{\epsilon A V^2}{2g^2}}_{F_e} - \underbrace{k(g_o - g)}_{F_{spring}}$$

What happens when I change g by a small increment dg?

↳ get an increment in the net attractive force F_{net}

$$dF_{net} = \frac{\partial F_{net}}{\partial g} dg = \underbrace{\left[-\frac{\epsilon A V^2}{g^3} + k\right]}dg$$

If $g\downarrow \rightarrow dg = (-)$, then for stability need $F_{net}\downarrow \rightarrow dF_{net} = (-)$

This must be (+)! → otherwise the plates collapse

Thus: $\boxed{k > \frac{\epsilon A V^2}{g^3}}$ (for a stable uncollapsed system)

Pull-in Voltage & Pull-in Gap

$V_{PI} \triangleq$ voltage @ which plates collapse

$g_{PI} \triangleq$ gap @ " " "

The plate goes unstable when:

$$k = \frac{\epsilon A V_{PI}^2}{g_{PI}^3} \quad (1)$$

$$F_{net} = 0 = \frac{\epsilon A V_{PI}^2}{2 g_{PI}^2} - k(g_o - g_{PI}) \quad (2)$$

Substitute (1) into (2):

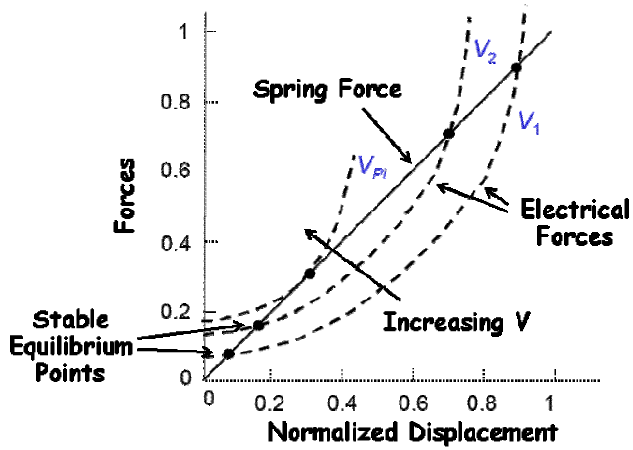
$$0 = \frac{\cancel{\epsilon A V_{PI}^2}}{2 \cancel{g_{PI}^2}} - \frac{\cancel{\epsilon A V_{PI}^2}}{\cancel{g_{PI}^3}}(g_o - g_{PI})$$

$$\frac{g_o - g_{PI}}{g_{PI}} = \frac{1}{2} \rightarrow g_o = \frac{3}{2} g_{PI}$$

$$\therefore \boxed{g_{PI} = \frac{2}{3} g_o}$$

When the gap is driven by a voltage to (2/3) the initial gap → collapse!

$$V_{PI} = \sqrt{\frac{k g_{PI}^3}{\epsilon A}} \rightarrow \boxed{V_{PI} = \sqrt{\frac{8}{27}\frac{k g_o^3}{\epsilon A}}}$$



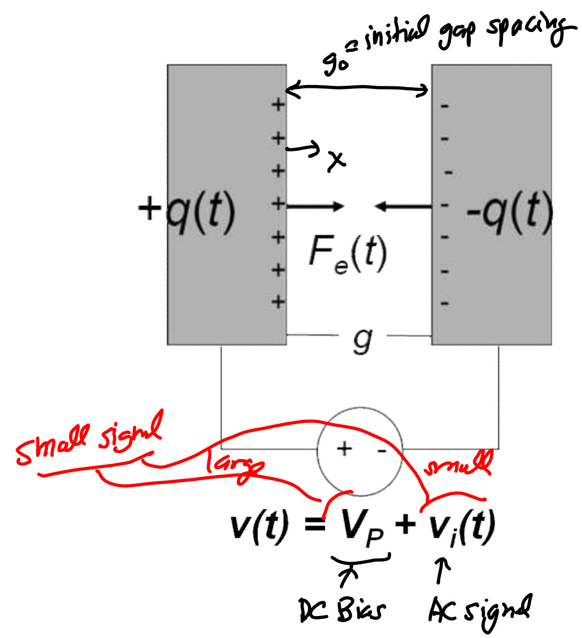
Advantages of Electrostatic Actuators:

- Easy to manufacture in micromachining processes, since conductors and air gaps are all that's needed → low cost!
- Energy conserving → only parasitic energy loss through I^2R losses in conductors and interconnects
- Variety of geometries available that allow tailoring of the relationships between voltage, force, and displacement
- Electrostatic forces can become very large when dimensions shrink → electrostatics scales well!
- Same capacitive structures can be used for both drive and sense of velocity or displacement
- Simplicity of transducer greatly reduces mechanical energy losses, allowing the highest Q's for resonant structures

Disadvantages of Electrostatic Actuators:

- Nonlinear voltage-to-force transfer function
- Relatively weak compared with other transducers (e.g., piezoelectric), but things get better as dimensions scale

Linearizing the Voltage-to-Force Transfer Function



$$F_e(t) = \frac{\partial W'}{\partial x} = \frac{\partial}{\partial x}\left[\frac{1}{2}C\,[v(t)]^2\right]$$

$$= \frac{1}{2}\frac{\partial C}{\partial x}[v(t)]^2 = \frac{1}{2}\frac{\partial C}{\partial x}[V_P + v_i(t)]^2$$

$$= \frac{1}{2}\left[V_P^2 + 2V_P v_i(t) + \cancel{[v_i(t)]^2}\right]\frac{\partial C}{\partial x}$$

$$\left[V_P >> v_i(t)\right] \Rightarrow F_e(t) = \underbrace{\frac{1}{2}V_P^2\frac{\partial C}{\partial x}}_{\text{DC offset}} + \underbrace{V_P\frac{\partial C}{\partial x}v_i(t)}_{\text{AC Drive Signal}}$$