Lecture Outline

* Reading: Senturia, Chpt. 10: §10.5, Chpt. 19
* Lecture Topics:
  - Estimating Resonance Frequency
  - Lumped Mass-Spring Approximation
  - ADXL-50 Resonance Frequency
  - Distributed Mass & Stiffness
  - Folded-Beam Resonator
Estimating Resonance Frequency

Clamped-Clamped Beam μResonator

\[ v_i = V_i \cos(\omega t) \rightarrow f_i = F_i \cos(\omega t) \]

- \( \omega \neq \omega_0 \): small amplitude
- \( \omega = \omega_0 \): maximum amplitude \( \rightarrow \) beam reaches its maximum potential and kinetic energies

\[ Q \approx 10,000 \]
Estimating Resonance Frequency

* Assume simple harmonic motion:

\[ x(t) = x_0 \cos(\omega t) \]

* Potential Energy:

\[ W(t) = \frac{1}{2} k x^2(t) = \frac{1}{2} k x_0^2 \cos^2(\omega t) \]

* Kinetic Energy:

\[ K(t) = \frac{1}{2} M x^2(t) = \frac{1}{2} M x_0^2 \omega^2 \sin^2(\omega t) \]

Estimating Resonance Frequency (cont)

* Energy must be conserved:

\[ \text{Potential Energy} + \text{Kinetic Energy} = \text{Total Energy} \]

Must be true at every point on the mechanical structure

Occurs at peak displacement

\[ W_{\text{max}} = \frac{1}{2} k x_0^2 = K_{\text{max}} = \frac{1}{2} M \omega^2 x_0^2 \]

Occurs when the beam moves through zero displacement

Maximum Potential Energy

Stiffness

Displacement Amplitude

Maximum Kinetic Energy

Mass

Radian Frequency

* Solving, we obtain for resonance frequency:

\[ \omega = \sqrt{\frac{k}{M}} \]
**Example: ADXL-50**

- The proof mass of the ADXL-50 is many times larger than the effective mass of its suspension beams
  - Can ignore the mass of the suspension beams (which greatly simplifies the analysis)
- Suspension Beam: \( L = 260 \mu m, h = 2.3 \mu m, W = 2 \mu m \)


**Lumped Spring-Mass Approximation**

- Mass is dominated by the proof mass
  - 60% of mass from sense fingers
  - Mass = \( M = 162 \text{ ng} \) (nano-grams)
- Suspension: four tensioned beams
  - Include both bending and stretching terms [A.P. Pisano, BSAC Inertial Sensor Short Courses, 1995-1998]
**ADXL-50 Suspension Model**

- Bending contribution:
  
  \[ k_b^{-1} = (1/k_o + 1/k_s) = 2 \left( \frac{(L/2)^3}{3E(Wh^3/12)} \right) = \frac{L^3}{EWh^3} = 4.2 \mu m / \mu N \]

- Stretching contribution:
  
  \[ k_s^{-1} = L/S = \frac{L}{\sigma_r Wh} = 1.14 \mu m / \mu N \]

- Total spring constant: add bending to stretching (since they are in parallel)
  
  \[ k_{st} = 4(k_b + k_s) = 4(0.24 + 0.88) = 4.5 \mu N / \mu m \]

**ADXL-50 Resonance Frequency**

- Using a lumped mass-spring approximation:
  
  \[ f' = \frac{1}{2\pi} \sqrt{\frac{k}{M}} = \frac{1}{2\pi} \sqrt{\frac{4.48N/m}{162x10^{-12} kg}} = 26.5 kHz \]

- On the ADXL-50 Data Sheet: \( f_o = 24 kHz \)
  
  - Why the 10% difference?
  - Well, it's approximate ... plus ...
  - Above analysis does not include the frequency-pulling effect of the DC bias voltage across the plate sense fingers and stationary sense fingers ... something we'll cover later on ...

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**Module 10: Resonance Frequency**
Distributed Mechanical Structures

* Vibrating structure displacement function:

\[ y(x, t) = \hat{y}(x) \cos(\omega t) \]

Maximum displacement function (i.e., mode shape function) \( \hat{y}(x) \)

Seen when velocity \( \dot{y}(x, t) = 0 \)

* Procedure for determining resonance frequency:
  - Use the static displacement of the structure as a trial function and find the strain energy \( W_{\text{max}} \) at the point of maximum displacement (e.g., when \( t=0, \pi/\omega, \ldots \))
  - Determine the maximum kinetic energy when the beam is at zero displacement (e.g., when it experiences its maximum velocity)
  - Equate energies and solve for frequency

Maximum Kinetic Energy

* Displacement: \( y(x, t) = \hat{y}(x) \cos[\omega t] \)

* Velocity: \( v(x, t) = \frac{\partial y(x, t)}{\partial t} = -\omega \hat{y}(x) \sin[\omega t] \)

* At times \( t = \pi/(2\omega), 3\pi/(2\omega), \ldots \)

\[ y(x, t) = 0 \]

Velocity topographical mapping

- The displacement of the structure is \( y(x, t) = 0 \)
- The velocity is maximum and all of the energy in the structure is kinetic (since \( W=0 \)):

\[ v(x, (2n+1)\pi/(2\omega)) = -\omega \hat{y}(x) \]
Maximum Kinetic Energy (cont)

- At times \( t = \pi/(2\omega), 3\pi/(2\omega), \ldots \)

\[
y(x, t) = 0
\]

**Velocity:** \( v(x, (2n + 1)\pi/(2\omega)) = -\omega \hat{y}(x) \)

\[
dK = \frac{1}{2} \cdot dm \cdot [v(x, t)]^2
\]

\[
dm = \rho(Wh \cdot dx)
\]

- Maximum kinetic energy:

\[
K_{\text{max}} = \int_0^L \frac{1}{2} \rho Wh \hat{y}^2(x, t') dx = \int_0^L \frac{1}{2} \rho Wh \omega^2 \hat{y}^2(x) dx
\]

The Raleigh-Ritz Method

- Equate the maximum potential and maximum kinetic energies:

\[
K_{\text{max}} = \int_0^L \frac{1}{2} \rho Wh \omega^2 \hat{y}^2(x) dx = W_{\text{max}}
\]

- Rearranging yields for resonance frequency:

\[
\omega = \sqrt{\frac{W_{\text{max}}}{\int_0^L \frac{1}{2} \rho Wh \hat{y}^2(x) dx}}
\]

- \( \omega \) = resonance frequency
- \( W_{\text{max}} \) = maximum potential energy
- \( \rho \) = density of the structural material
- \( W \) = beam width
- \( h \) = beam thickness
- \( \hat{y}(x) \) = resonance mode shape
Example: Folded-Beam Resonator

- Derive an expression for the resonance frequency of the folded-beam structure at left.

Use Rayleigh-Ritz method.

\[ KE_{\text{max}} = PE_{\text{max}} \]

Kinetic Energy:

\[ KE_{\text{max}} = KE_F + KE_E + KE_b \]

- shuttle truss beams

- mass of both beams

Must integrate since the beam velocity is a function of location y!

Get Kinetic Energies

- Folded-beam suspension

- Shuttle w/ mass \( M_s \)

- Folding truss w/ mass \( M_t \)

- Anchor \( h = \text{thickness} \)

Velocity of the shuttle: \( N_F = \omega_0 \sum_o \)

Resonance Frequency

Maximum Displacement Amplitude

\[ KE_F = \frac{1}{2} M_s \omega_0^2 \]

\[ KE_E = \frac{1}{2} \sum_o \omega_0^2 M_s \]

Velocity of the truss: \( N_t = \frac{1}{2} \omega_0 \)

\[ KE_E = \frac{1}{2} \left( \sum_o \omega_0^2 \right) M_t \]

Velocity of the beam segments:

- Assume the mode shape is the same as the static displacement shape

- For segment AB:

\[ \alpha(y) = \frac{F_x}{4PE_E} \left( 3y^2 - 2y^3 \right) \quad 0 \leq y \leq L \]
Folded-Beam Suspension

Comb-Driven Folded Beam Actuator

\[
\frac{F_x}{4EI} \left( 3y^2 - 2y^3 \right) \quad 0 \leq y \leq L
\]

\[
F_{x,y} = 0
\]

\[
F_{x,y} \left( L - y \right) = \frac{12EIz}{L^3} = \frac{L}{2}
\]

Get Kinetic Energies (cont)

At \( y = L \):

\[
\frac{X_0}{2} \left[ \frac{3}{2} \left( \frac{A}{L} \right)^2 - \frac{3}{2} \left( \frac{B}{L} \right)^2 \right]
\]

Which yields the velocity:

\[
\frac{v_y(y)}{AB} = \frac{X_0}{2} \left[ \frac{3}{2} \left( \frac{A}{L} \right)^2 - \frac{3}{2} \left( \frac{B}{L} \right)^2 \right] \omega_y
\]

Plugging into the expression for \( K_{E_b} \):

\[
K_{E_b}(AB) = \frac{1}{2} \int_0^L \frac{X_0^2 \omega_y^2}{4} \left[ \frac{3}{2} \left( \frac{A}{L} \right)^2 - \frac{3}{2} \left( \frac{B}{L} \right)^2 \right] dy
\]

Static mass of beam (AB)

\[
= \frac{X_0^2 \omega_y^2}{4L} \int_0^L \left[ \frac{3}{2} \left( \frac{A}{L} \right)^2 - \frac{3}{2} \left( \frac{B}{L} \right)^2 \right] dy
\]

\[
K_{E_b}(AB) = \frac{13}{128} X_0^2 \omega_y^2 M(AB)
\]
Get Kinetic Energies (cont)

For segment CD:
\[ \phi_C = X_0 \left( \frac{1}{2} \left( \frac{h}{L} \right)^2 + \left( \frac{h}{L} \right)^3 \right) \]

Thus:
\[ K_{E(CD)} = \frac{X_0^2 \omega_0^2 M_{(CD)}}{2L} \int_0^L \left[ \frac{1}{2} \left( \frac{h}{L} \right)^2 + \left( \frac{h}{L} \right)^3 \right] dh \]

\[ K_{E(CD)} = \frac{2 \pi^2}{280} X_0^2 \omega_0^2 M_{(CD)} \]

Static mass of beam [CD]

Let \( M_b \) be total mass of the \( p \) beams.

Then: \( M_{[AS]} = M_{(CD)} = \frac{1}{p} M_b \)

Thus:
\[ K_E = 4K_{E(AB)} + 4K_{E(CD)} = \frac{6}{35} X_0^2 \omega_0^2 M_b \]

and
\[ K_{E_{max}} = X_0^2 \omega_0^2 \left( \frac{1}{2} M_s + \frac{1}{p} M_b + \frac{6}{35} M_b \right) \]

Get Potential Energy & Frequency

PE\(_{max}\) is simply the work done due to achieve maximum deflection:
\[ PE_{\max} = \frac{1}{2} k x^2 \]

\[ PE_{\max} = \frac{1}{2} k x_{\max}^2 \]

Thus, using Raleigh-Ritz:
\[ K_{E_{max}} = PE_{\max} \]

\[ X_0^2 \omega_0^2 \left( \frac{1}{2} M_s + \frac{1}{p} M_b + \frac{6}{35} M_b \right) = \frac{1}{2} k x_{\max}^2 \]

\[ \omega_0 = \left( \frac{k_p}{M_{eq}} \right)^{1/2} \]

Where \( M_{eq} = M_s + \frac{1}{p} M_b + \frac{18}{35} M_b \)

(Resonance Frequency of a Folded-Beam Suspended Shuttle)
Brute Force Methods for Resonance Frequency Determination

Basic Concept: Scaling Guitar Strings

Guitar String

Vibrating “A” String (110 Hz)

Freq. Equation:

$$f_0 = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$$

Mechanical Resonator

- Metallized Electrode
- Anchor
- Polysilicon Clamped-Clamped Beam

Performance:

- $$f_r = 8.5 \text{MHz}$$
- $$Q_{vac} = 8,000$$
- $$Q_{air} \sim 50$$
- $$L_r = 40.8 \mu m$$
- $$m_r \sim 10^{-13} \text{ kg}$$
- $$W_r = 8 \mu m$$, $$h_r = 2 \mu m$$
- $$d = 1000 \AA$$, $$V_r = 5 \text{ V}$$
- Pressure = 70 mTorr

[Bannon 1996]
Anchor Losses

Problem: direct anchoring to the substrate ⇒ anchor radiation into the substrate ⇒ lower Q

Solution: support at motionless nodal points ⇒ isolate resonator from anchors ⇒ less energy loss ⇒ higher Q

92 MHz Free-Free Beam μResonator

- Free-free beam μmechanical resonator with non-intrusive supports ⇒ reduce anchor dissipation ⇒ higher Q

Design/Performance:
- $L=13.1\mu m$, $W=8\mu m$
- $t=2\mu m$, $d=1000\AA$
- $V_p=29-76V$, $W_p=2.8\mu m$
- $f_0=92.25\text{MHz}$
- $Q=7450$ @ 10mTorr

[Wang, Yu, Nguyen 1998]
Higher Order Modes for Higher Freq.

2nd Mode Free-Free Beam

3rd Mode Free-Free Beam

Distinct Mode Shapes

Electrodes

Anchor

Support Beam

$L_e = 20.3 \, \mu m$

$h = 2.1 \, \mu m$

Frequency [MHz]

Transmission [dB]

Q = 11.500

Frequency [MHz]

Transmission [dB]

Phase [degree]

Flexural-Mode Beam Wave Equation

Transverse Displacement

W = width

$u = \frac{\partial^2 u}{\partial t^2}$

ma

Combining (1) & (2):

$\frac{\partial^2 M}{\partial x^2} = -\rho A \frac{\partial^2 u}{\partial t^2}$

$\frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 u}{\partial x^2} \right) = -\rho A \frac{\partial^2 u}{\partial t^2}$

$I_g = \frac{Wh^3}{12}$

• Derive the wave equation for transverse vibration:

Dynamic Equilibrium Condition for Forces in the y-direction:

$F - (F + \frac{\partial F}{\partial x} dx) - \rho A dx \frac{\partial u}{\partial t} = 0$ (1) neglects $\frac{\partial F}{\partial x} dx$ term

and No moment equilibrium condition:

$-F dx + \frac{\partial M}{\partial x} dx \approx 0$ (2)
Example: Free-Free Beam

* Determine the resonance frequency of the beam
* Specify the lumped parameter mechanical equivalent circuit
* Transform to a lumped parameter electrical equivalent circuit

* Start with the flexural-mode beam equation:

\[
\frac{\partial^2 u}{\partial t^2} = \left(\frac{EI}{\rho A}\right) \frac{\partial^4 u}{\partial x^4}
\]

Free-Free Beam Frequency

* Substitute \( u = u_1 e^{j\omega t} \) into the wave equation:

\[
\frac{\partial^4 u}{\partial x^4} = \left(\frac{\omega^2 \rho A}{EI}\right) u
\]

(1)

* This is a 4th order differential equation with solution:

\[
u(x) = A \cosh kx + B \sinh kx + C \cos kx + D \sin kx \quad (2)
\]

* Boundary Conditions:

\[
\begin{align*}
\text{At } x &= 0 \\
\text{At } x &= L
\end{align*}
\]

\[
\begin{align*}
\frac{\partial^2 u}{\partial x^2} &= 0 \\
\frac{\partial^4 u}{\partial x^4} &= 0 \\
\frac{\partial^3 u}{\partial x^3} &= 0 \\
\frac{\partial^4 u}{\partial x^4} &= 0 \\
M &= 0 \text{ (Bending moment)} \\
\frac{\partial M}{\partial x} &= 0 \text{ (Shearing force)}
\end{align*}
\]
Free-Free Beam Frequency (cont)

* Applying B.C.'s, get $A=C$ and $B=D$, and

\[
\begin{bmatrix}
(cosh \ k \ell - cos \ k \ell) & (sinh \ k \ell - sin \ k \ell) \\
(sinh \ k \ell + sin \ k \ell) & (cosh \ k \ell - cos \ k \ell)
\end{bmatrix}
\begin{bmatrix}
A \\
B
\end{bmatrix} = 0 \tag{3}
\]

* Setting the determinant $= 0$ yields

\[
cos k \ell = \frac{1}{cosh k \ell}
\]

* Which has roots at

\[
k_1 \ell = 4.730 \quad k_2 \ell = 7.853 \quad k_3 \ell = 10.996
\]

* Substituting (2) into (1) finally yields:

\[
k^4 = \frac{pA}{EI} \omega^2 \quad \therefore \quad f_o = \frac{(k_w \ell)^2}{2 \pi c^2} \sqrt{\frac{EI}{pA}}
\]

Higher Order Free-Free Beam Modes

<table>
<thead>
<tr>
<th>Mode</th>
<th>n</th>
<th>Nodal Points</th>
<th>$k_n \ell$</th>
<th>$f_o / f_0$</th>
</tr>
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<tbody>
<tr>
<td>Fundamental</td>
<td>1</td>
<td>2</td>
<td>4.730</td>
<td>1.000</td>
</tr>
<tr>
<td>1st Harmonic</td>
<td>2</td>
<td>3</td>
<td>7.853</td>
<td>2.757</td>
</tr>
<tr>
<td>2nd Harmonic</td>
<td>3</td>
<td>4</td>
<td>10.996</td>
<td>5.404</td>
</tr>
<tr>
<td>3rd Harmonic</td>
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<td>5</td>
<td>14.137</td>
<td>8.932</td>
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<tr>
<td>4th Harmonic</td>
<td>5</td>
<td>6</td>
<td>17.279</td>
<td>13.344</td>
</tr>
</tbody>
</table>

\[\text{More than 10x increase}\]

Fundamental Mode ($n=1$)

$1^{st}$ Harmonic ($n=2$)

$2^{nd}$ Harmonic ($n=3$)
Mode Shape Expression

- The mode shape expression can be obtained by using the fact that \( A=C \) and \( B=D \) into (2), yielding

\[
\psi_n = \mathcal{N} \left[ \left( \frac{\alpha}{\beta} \right) \left( \cosh kx + \cos kx \right) + \left( \sinh kx + \sin kx \right) \right]
\]

- Get the amplitude ratio by expanding (3) [the matrix] and solving, which yields

\[
\frac{\alpha}{\beta} = \frac{\sin k\ell - \sinh k\ell}{\cosh k\ell - \cos k\ell}
\]

- Then just substitute the roots for each mode to get the expression for mode shape

Fundamental Mode (n=1)

[Substitute \( k_{n1} \ell = 4.730 \)]