Lecture Outline

- Reading: Senturia, Chpt. 5
- Lecture Topics:
  - Lumped Mass
  - Lumped Stiffness
  - Lumped Damping
  - Lumped Mechanical Equivalent Circuits
  - Electromechanical Analogies
Lumped Parameter Mechanical Equivalent Circuit

Equivalent Dynamic Mass

- Once the mode shape is known, the lumped parameter equivalent circuit can then be specified.
- Determine the equivalent mass at a specific location \( x \) using knowledge of kinetic energy and velocity.

Maximum Kinetic Energy

\[ \text{Maximum Kinetic Energy} = \frac{1}{2} m v_x^2 \]

Equivalent Mass

\[ M_{eqx} = \frac{K.E.}{\frac{1}{2} v_x^2} \]

Maximum Velocity @ location \( x \)

\[ \text{Maximum Velocity Function} = \frac{1}{2} L \int_0^L v^2(x) \, dx \]
Equivalent Dynamic Mass

* For the folded-beam structure, we've already determined the maximum kinetic energy.
* And in our resonance frequency analysis, we've already determined expressions for velocity.

\[
\begin{align*}
K_{\text{eq}(\text{truss})} &= \frac{1}{2} \frac{V_{\text{max}}^2}{X_{\text{max}}} \\
M_{\text{eq}(\text{truss})} &= \frac{\omega_0^2 X_{\text{max}} (M_T + \frac{1}{2} M_L + \frac{12}{35} M_b)}{\omega_0^2}
\end{align*}
\]

Equivalent Dynamic Stiffness & Damping

* Stiffness then follows directly from knowledge of mass and resonance frequency.

\[
\omega_0 = \sqrt{\frac{K_{\text{eq}}(x)}{M_{\text{eq}}(x)}} \quad \Rightarrow \quad K_{\text{eq}}(x) = \omega_0^2 M_{\text{eq}}(x)
\]

* And damping also follows readily from knowledge of Q or other loss measurands.

\[
Q = \frac{\omega_0 M_{\text{eq}}(x)}{C_{\text{eq}}(x)} \quad \Rightarrow \quad C_{\text{eq}}(x) = \frac{\omega_0 M_{\text{eq}}(x)}{Q} \cdot \frac{K_{\text{eq}}(x) M_{\text{eq}}(x)}{Q}
\]

* With mass, stiffness, and damping \( \Rightarrow \) lumped parameter equivalent circuit.
Get Potential Energy & Frequency

Folded-beam suspension

Anchor \( h = \text{thickness} = 2 \mu m \)

- Shuttle w/ mass \( M_s \)
- Folding truss w/ mass \( M_t \)

\[ M_{eq(shuttle)} = 8.64 \times 10^{-11} \text{ kg} \]
\[ K_{eq(shuttle)} = 4.8 \text{ N/m} \]
\[ C_{eq(shuttle)} = 4.08 \times 10^{-10} \text{ kg/s} \]

\[ M_{eq(truss)} = 2.16 \times 10^{-11} \text{ kg} \]
\[ K_{eq(truss)} = 19.2 \text{ N/m} \]
\[ C_{eq(truss)} = 1.02 \times 10^{-10} \text{ kg/s} \]

Electromechanical Analogies

\[ F(t) = F \cos(\omega t) \rightarrow \dot{x}(t) = \dot{X} \cos(\omega t) \]

Equation of Motion:

\[ m_{eq} \ddot{x} + c_{eq} \dot{x} + k_{eq} x = F(t) \]

\[ \Rightarrow \text{using phasor concepts:} \]

\[ F = j \omega m_{eq} \dot{X} + \frac{k_{eq}}{j \omega} X + c_{eq} \dot{X} \]

\[ \Rightarrow \text{by analogy:} \]

\[ F \rightarrow N \]
\[ \dot{x} \rightarrow \dot{I} \]
\[ x \rightarrow i \]
\[ k_{eq} \rightarrow r_x \]
\[ c_{eq} \rightarrow j \omega L_x \]

[Parameter Relationships in the Current Analogy]
Electromechanical Analogies (cont)

- Mechanical-to-electrical correspondence in the current analogy:

<table>
<thead>
<tr>
<th>Mechanical Variable</th>
<th>Electrical Variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>Damping, $c$</td>
<td>Resistance, $R$</td>
</tr>
<tr>
<td>Stiffness$^{-1}$, $k^{-1}$</td>
<td>Capacitance, $C$</td>
</tr>
<tr>
<td>Mass, $m$</td>
<td>Inductance, $L$</td>
</tr>
<tr>
<td>Force, $f$</td>
<td>Voltage, $V$</td>
</tr>
<tr>
<td>Velocity, $v$</td>
<td>Current, $I$</td>
</tr>
</tbody>
</table>

Bandpass Biquad Transfer Function

\[
F(s) = \frac{X(s)}{E(s)} = \frac{\frac{k_{EQ}}{s+\frac{1}{\omega_0}}} {1 + \frac{s}{\frac{1}{\omega_0}}} \\
X(s) = E(s) \frac{k_{EQ}}{s+\frac{1}{\omega_0}}
\]
3CC 3\lambda/4 Bridged \mu Mechanical Filter

Performance:
- \( f_0 = 9 \text{MHz} \)
- \( BW = 20 \text{kHz} \)
- \( PBW = 0.2\% \)
- \( I.L. = 2.79 \text{dB} \)
- \( \text{Stop. } \text{Rej.} = 51 \text{dB} \)
- \( 20\text{dB S.F.} = 1.95, \ 40\text{dB S.F.} = 6.45 \)

Design:
- \( L_r = 40 \mu m \)
- \( W_r = 6.5 \mu m \)
- \( h_r = 2 \mu m \)
- \( L_c = 3.5 \mu m \)
- \( L_p = 1.6 \mu m \)
- \( V_p = 10.47 \text{V} \)
- \( P = -5 \text{dBm} \)
- \( R_Q = R_Q' = 12k\Omega \)

Micromechanical Filter Circuit

\( R_Q \) Input
\( v_i \)
\( V_p \) Output
\( v_o \)
\( R_Q \)

\( 3\lambda/4 \) Bridging Beam
\( \lambda/4 \) Coupling Beam
\( \lambda/4 \) Resonator

\( \eta_c, \eta_t, \eta_{\eta} \)
\( m, \frac{1}{k}, c \)

\( \text{Bridging Beam} \)
\( \text{Coupling Beam} \)
\( \text{Resonator} \)

\( 1: \eta_c, \frac{1}{k}, c \)
\( \eta_t \)
\( \eta_{\eta} : \eta_c, \frac{1}{k}, c \)

\( [S.-S. Li, Nguyen, FCS'05] \)

\( [Li, \ et \ al., \ UFFCS'04] \)
Micromechanical Filter Circuit

Input

Output

R_Q

v_i

v_o

| R_P |

1/\eta_e

1/\eta_c

1/\eta_b

1/

\omega

V_P

Micromechanical Filter Circuit

Input

Output

R_Q

v_i

v_o

| R_P |

1/\eta_e

1/\eta_c

1/\eta_b

1/

\omega

V_P

Micromechanical Filter Circuit

Input

Output

R_Q

v_i

v_o

| R_P |

1/\eta_e

1/\eta_c

1/\eta_b

1/

\omega

V_P

Micromechanical Filter Circuit

Input

Output

R_Q

v_i

v_o

| R_P |

1/\eta_e

1/\eta_c

1/\eta_b

1/

\omega

V_P
Micromechanical Filter Circuit

All circuit element values determined by CAD layout

Amenable to automated circuit generation

3CC 3λ/4 Bridged μMechanical Filter

Performance:
\[ f_0 = 9 \text{MHz}, \quad BW = 20 \text{kHz}, \quad PBW = 0.2\% \]
\[ I.L. = 2.79 \text{dB}, \quad \text{Stop. Rej.} = 51 \text{dB} \]
\[ 20 \text{dB S.F.} = 1.95, \quad 40 \text{dB S.F.} = 6.45 \]

Design:
\[ L_r = 40 \mu\text{m}, \quad W_r = 6.5 \mu\text{m}, \quad h_r = 2 \mu\text{m}, \quad L_c = 3.5 \mu\text{m}, \quad L_b = 1.6 \mu\text{m} \]
\[ V_p = 10.47 \text{V}, \quad P = 5 \text{dBm}, \quad R_{Qe} = 12 \text{kΩ} \]

8.7 \quad 8.9 \quad 9.1
Frequency [MHz]

[S.-S. Li, Nguyen, FCS'05]

Li, et al., UFFCS'04

[In, Out]
Beam Resonator Equivalent Circuits
(Pretty Much the Same Stuff)

Equivalent Dynamic Mass

- Once the mode shape is known, the lumped parameter
  equivalent circuit can then be specified
- Determine the equivalent mass at a specific location \( x \) using
  knowledge of kinetic energy and velocity

\[ M_{eqx} = \frac{K.E.}{\frac{1}{2}v_x^2} = \frac{\frac{1}{2}m \int_0^l v^2(x) \, dx}{\frac{1}{2}v_x^2} \]

Maximum Kinetic Energy

Maximum Velocity @ location \( x \)

Maximum Velocity Function

Location \( x \)
Equivalent Dynamic Mass

- We know the mode shape, so we can write expressions for displacement and velocity at resonance

\[ u(x) = B \left[ 9(\cosh kx + \cos kx' + \sinh kx + \sin kx') \right], \quad S = \frac{A}{b} \]

\[ \left[ V(x) = \omega^2 u(x) \right] \Rightarrow M_{eq}(x) = \frac{\kappa_{max} E}{2[E]} \int_a^b \frac{1}{\rho A} \frac{\omega^2}{2} \left[ u(x') \right]^2 dx' \]

\[ M_{eq}(x) = \frac{\rho A L}{b} \frac{\omega^2}{2} \frac{\left( 9(\cosh kx + \cos kx' + \sinh kx + \sin kx') \right)^2}{\left( 9(\cosh kx + \cos kx' + \sinh kx + \sin kx') \right)^2} \]

Equivalent Dynamic Stiffness & Damping

- Stiffness then follows directly from knowledge of mass and resonance frequency

\[ \omega_0^2 \frac{K_{eq}(x)}{M_{eq}(x)} \rightarrow K_{eq}(x) = \omega_0^2 M_{eq}(x) \]

- And damping also follows readily

\[ Q = \frac{\omega_0 M_{eq}(x)}{C_{eq}(x)} \rightarrow C_{eq}(x) = \frac{\omega_0 M_{eq}(x)}{Q} = \frac{\sqrt{K_{eq}(x) M_{eq}(x)}}{Q} \]
**Equivalent Lumped Mechanical Circuit**

Location $x$

\[ K_{eq}(x) = \omega_0^2 M_{eq}(x) \]

\[ M_{eq}(x) = \frac{\rho A}{\omega_0} \int_0^x [u(x')]^2 \, dx' \]

\[ C_{eq}(x) = \frac{\omega_0 M_{eq}(x)}{Q} \]

**Example:** Polysilicon with $\ell=14.9\mu m$, $W=6\mu m$, $h=2\mu m \rightarrow 70$ MHz

\[ K_{eq}(0) = 19,927 \text{ N/m} \]

\[ M_{eq}(0) = 1.03 \times 10^{-13} \text{ kg} \]

\[ C_{eq}(0) = 5.66 \times 10^{-9} \text{ kg/s} \]

\[ K_{eq}(l/2) = 53,938 \text{ N/m} \]

\[ M_{eq}(l/2) = 2.78 \times 10^{-13} \text{ kg} \]

\[ C_{eq}(l/2) = 1.53 \times 10^{-8} \text{ kg/s} \]