Lecture Outline

- Reading: Senturia, Chpt. 6, Chpt. 14
- Lecture Topics:
  - Input Modeling
    - Input Equivalent Ckt.
  - Current Modeling
    - Output Current Into Ground
    - Input Current
    - Complete Electrical-Port Equiv. Ckt.
  - Impedance & Transfer Functions
Input Modeling

Electromechanical Analogies

Equation of Motion:
\[ m_{eq} \ddot{x} + c_{eq} \dot{x} + k_{eq} x = F(t) \]

\[ N(t) \rightarrow \text{Voltage} \]

Impedance looking in:
\[ \frac{N}{i} = \frac{jω i + \frac{i}{jω} + r_x}{1 + jω L + \frac{jω}{C} + r_x} \]

\[ N = jω L i + \frac{ω C}{jω} i + r_x i \]

Parameter Relationships in the Current Analogy:
\[ F \rightarrow N \quad m_{eq} \rightarrow l_x \quad c_{eq} \rightarrow r_x \]
\[ x \rightarrow i \quad k_{eq} \rightarrow \frac{1}{C_x} \]
Bandpass Biquad Transfer Function

\[ F(s) = \frac{X(j\omega)}{s^2 + \omega_0^2 + \frac{Q}{Q_0}s + \omega_0^2} \]

Force-to-Velocity Relationship

- The relationship between input voltage \( v_i \) and force \( F_{d1} \):
  \[ F_{d1} \approx -V_P \frac{\partial C_i}{\partial x} v_i \]

- When displacement \( x \) is the mechanical output variable:
  \[ \frac{X(s)}{F_{d1}(s)} = \frac{1}{k s^2 + (\omega_0/Q)s + \omega_0^2} \]

- When velocity \( v \) is the mechanical output variable:
  \[ \frac{v(s)}{F_{d1}(s)} = \frac{sX(s)}{F_{d1}(s)} = \frac{1}{k s^2 + (\omega_0/Q)s + \omega_0^2} \]
**Force-to-Velocity Equiv. Ckt.**

- Combine the previous lumped LCR mechanical equivalent circuit with a circuit modeling the capacitive transducer → circuit model for voltage-to-velocity.

**Equiv. Circuit for a Linear Transducer**

- A transducer ... 
  - converts energy from one domain (e.g., electrical) to another (e.g., mechanical)
  - has at least two ports
  - is not generally linear, but is virtually linear when operated with small signals (i.e., small displacements)
**Equiv. Circuit for a Linear Transducer**

For physical consistency, use a transformer equivalent circuit to model the energy conversion from the electrical domain to mechanical domain.

\[
\begin{bmatrix}
  e_2 \\
  f_2
\end{bmatrix} = \begin{bmatrix}
  \eta & 0 \\
  0 & -\frac{1}{\eta}
\end{bmatrix} \begin{bmatrix}
  e_1 \\
  f_1
\end{bmatrix}
\]

Describing Matrix

**Electromechanical Equivalent Circuit**

- \( e_2 = F_{d1} \), \( e_1 = v_1 \), just need \( \eta_1 \):
- From the matrix: \( e_2 = \eta e_1 \)

\[
F_{d1} \approx -V_P \frac{\partial C_1}{\partial x} v_1 \rightarrow \eta_1 = V_P \frac{\partial C_1}{\partial x}
\]
Output Modeling

Output Current Into Ground

• When the mass moves with time-dependent displacement $x(t)$, the electrode-to-mass capacitors $C_1(x,t)$ and $C_2(x,t)$ vary with time.

• This generates an output current:

$$i_1 = \frac{d}{dt} \left[ \frac{q}{C} \right] + C_1 \frac{d^2 x}{dt^2} + V \frac{d^2 x}{dt^2}$$

$$i_2(t) = C_2(x,t) \frac{dV_1(t)}{dt} + V_1(t) \frac{dC_2(x,t)}{dt}$$

$$\left[ V(t) - V_P \right] = I_2 - V_P \frac{dC_2}{dt} - V_P \frac{dC_2}{dx} \frac{dx}{dt}$$

In phasor form:

$$I_2(j\omega) = -j\omega V_P \frac{dC_2}{dx} x$$
Output Current Into Ground

Again, model with a transformer:

\[ I_2(j\omega) = -j\omega V_p \frac{\partial c_2}{\partial x} x = -V_p \frac{\partial c_2}{\partial x} U \]

90° phase lag

\[ \phi(\omega) \rightarrow I_2 = 0 \text{ when } x = 0 \]

\[ f_x = \frac{1}{\eta_2} f_1 \rightarrow f_2 = -\eta_2 f_1 \]

\[ f_1 \rightarrow I_2; f_2 : U \Rightarrow I_2 = -\eta U \]

\[ x \times \frac{\partial f_x}{\partial x} \]

Input Current Expression

Get \( I_c(j\omega) \):

\[ i_c(j\omega) = C_1 \frac{dx}{dt} + V_1 \frac{dC_1}{dx} \frac{x}{dt} \]

\[ [V_1 \cdot \frac{dx}{dt} - V_p] \rightarrow i_c = C_1 \frac{dx}{dt} + [V_1 - V_p] \frac{dx}{dt} - x \frac{dx}{dt} \]

\[ \Rightarrow I_c(j\omega) = C_1 V_1 \frac{dx}{dt} + jwV_1 \frac{dx}{dt} = \frac{x}{dx} \]

\[ [V_1 \ll V_p] \Rightarrow I_c(j\omega) = jwC_1 V_1 - jwV_p \frac{dx}{dx} \]

\[ \text{Feed-through Current} \quad \text{Motional Current} \]

\[ \phi(\omega) \rightarrow \text{due to mass motion} \]

\[ \text{at DC: } x = \frac{F_{dc}}{k} = \frac{1}{k} V_p \frac{\partial c_2}{\partial x} V_i \]

\[ \text{at resonant: } \frac{\omega}{\phi} = \frac{\phi_{dc}}{j} = \frac{1}{j \omega V_p \frac{\partial c_2}{\partial x} V_i} \]

\[ \phi(\omega) \rightarrow \text{90° phase lag} \]
**Input Current Expression (cont)**

Thus: @ resonance

\[ I_1(j\omega) = j\omega C_1 V_1 + j\omega_0 \left( \frac{\partial C_1}{\partial x} \right) \frac{Q}{jk} V_1 \]

\[ = j\omega C_1 V_1 + \omega_0 \frac{Q}{k} n_1^2 V_1 \]

+ 90° phase shifted from \( V_1 \)

This is a capacitance in shunt with the input to Electrode 1

- \( C_2 \):

Motional Resistance:

\[ R_{x1} V_1 \frac{d}{dt} = \frac{k}{\omega_0 n_1^2} \]

\[ \Rightarrow \eta_{e1} = \frac{C_{ol}}{d_1} \]

The equivalent ckt. behavior gets this right!

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**Complete Electrical-Port Equiv. Circuit**

Static electrode-to-mass overlap capacitance

\[ \eta_{e1} = V_p \frac{\partial C_1}{\partial x} = V_p \frac{C_{ol}}{d_1} \]

\[ \eta_{e2} = V_p \frac{\partial C_2}{\partial x} = V_p \frac{C_{ol}}{d_2} \]

\[ l_x = m \]

\[ c_x = \frac{1}{k} \]

\[ r_x = b \]
Input Impedance Into Port 1

What is the impedance seen looking into port 1 with port 2 shorted to ground?

From our transformer model:

\[
\begin{align*}
\frac{e_1}{i_1'} &= \frac{1}{\eta} \left[ \begin{array}{c} e_2 \\ f_1 \\ \eta_1 \\ \eta_2 \\ f_2 \end{array} \right] \\
&= \frac{1}{\eta} \left[ \begin{array}{c} \eta e_1 \\ f_1 \\ \eta_1 e_1 \\ \eta_2 e_1 \\ f_2 \end{array} \right] \\
&= \frac{1}{\eta} \cdot \frac{1}{\eta_2} \left( j \omega C_x + \frac{1}{j \omega C_x} + r_x \right) \\
&= j \omega \left( \frac{L_x}{\eta^2} \right) + \frac{1}{j \omega (\eta_2 C_x)} + \frac{r_x}{\eta_2} \frac{1}{R_{x2}}.
\end{align*}
\]

Input Impedance Into Port 2

What is the impedance seen looking into port 2 with port 1 shorted to ground?

\[
\begin{align*}
\frac{e_2}{i_2'} &= 2 \cdot \frac{1}{\eta_2} \left( j \omega L_x + \frac{1}{j \omega C_x} + r_x \right) \\
&= j \omega \left( \frac{L_x}{\eta^2} \right) + \frac{1}{j \omega (\eta_2 C_x)} + \frac{r_x}{\eta_2} \frac{1}{R_{x2}}.
\end{align*}
\]

Note: They are not the same as \( L_x, C_x \), \( R_{x2} \) on Port 1.
**Port 1 to 2 TransG Across the Circuit**

What is the transconductance from port 1 to port 2 with port 2 shorted to ground?

\[ \eta_e^{1 \to 2} = \frac{I_2}{V_2} \]

\[ \eta_e^{1 \to 2} = \frac{1}{R_x} \left\{ \frac{1}{\eta_{c1}} \left[ \frac{1}{\eta_{c1}} + \frac{1}{\eta_{c2}} + j\omega C \right] \right\} \]

**Port 1 to 2 \( v_1 \)-to-\( i_o \) Transfer Function**

\[ \frac{I_o}{V_1}(s) = \frac{1}{sC_x + \frac{1}{sC_x} + R_x} \]

Separate freq response & magnitude:

\[ \frac{I_o}{V_1}(s) = \frac{sC_x}{s^2C_x + s(\omega_0/R_x) + \omega_0^2} \]

\[ \frac{1}{sC_x} = \omega_0^2, \quad Q = \frac{\omega_0}{R_x}, \quad R_x = \frac{1}{sC_x} \]

\[ \frac{I_o}{V_1}(s) = \frac{\omega_0^2}{s^2 + s(\omega_0/R_x) + \omega_0^2} \]

\[ \Re(s) = \frac{\omega_0^2}{s^2 + s(\omega_0/R_x) + \omega_0^2} \]

\[ \Im(s) = \left[ \frac{\omega_0^2}{s^2 + s(\omega_0/R_x) + \omega_0^2} \right] \]

\[ \omega_0 = \Re(s) = 0 \]

\[ s = j\omega_0 \Rightarrow \Im(s) = \omega_0 \]

\[ s = 0 \Rightarrow \Re(s) = 0 \]

Gain Bandpass Biquad

This will always be the same.

Thus, could just work @ resonant frequency and just multiply by \( \Re(s) \).
Condensed Equiv. Circuit (Symmetrical)

Holds for the symmetrical case, where port 1 and port 2 are identical.

If \( \eta_1 = \eta_2 \), then ...

\[
\begin{align*}
L_x &= \frac{m}{\eta_e^2} \\
C_x &= \frac{\eta_e^2}{k} \\
R_x &= \frac{b}{\eta_e^2}
\end{align*}
\]

Phasings of Signals

*Below: plots of resonance electrical and mechanical signals vs. time, showing the phasings between them*