EE C247B – ME C218
Introduction to MEMS Design
Spring 2018

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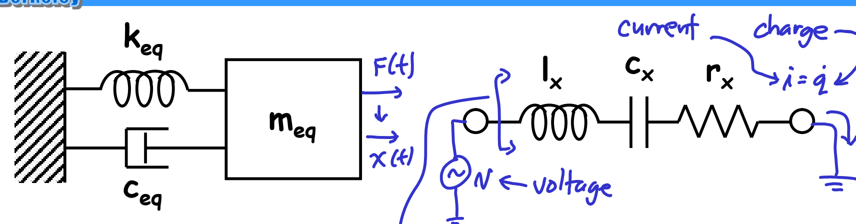
Lecture Module 13: Equivalent Circuits II

Lecture Outline

- Reading: Senturia, Chpt. 6, Chpt. 14
- Lecture Topics:
  - Input Modeling
    - Force-to-Velocity Equiv. Ckt.
    - Input Equivalent Ckt.
  - Current Modeling
    - Output Current Into Ground
    - Input Current
    - Complete Electrical-Port Equiv. Ckt.
  - Impedance & Transfer Functions

Input Modeling

Electromechanical Analogies



$F(t) = F\cos(\omega t) \rightarrow x(t) = X\cos\omega t$

Equation of Motion:

$$m_{eq}\ddot{x} + c_{eq}\dot{x} + k_{eq}x = F(t)$$

⇒ using phasor concepts:

$$F = j\omega m_{eq}\dot{x} + \frac{k_{eq}}{j\omega}\dot{x} + c_{eq}\dot{x}$$

$v(t) = V\cos\omega t \rightarrow i(t) = I\cos\omega t$

Impedance looking in:

$$\frac{v}{i} = j\omega l_x + \frac{1}{j\omega C_x} + r_x$$

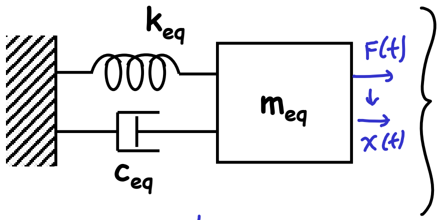
$$v = j\omega l_x i + \frac{(1/C_x)}{j\omega} i + r_x i$$

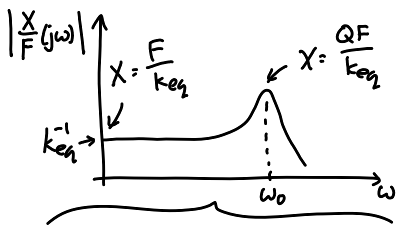
⇒ by analogy:

$$\boxed{F \rightarrow v \quad \dot{x} \rightarrow i \quad m_{eq} \rightarrow l_x \quad k_{eq} \rightarrow \frac{1}{C_x} \quad c_{eq} \rightarrow r_x}$$

[Parameter Relationships in the Current Analogy]

Bandpass Biquad Transfer Function





$$\boxed{\frac{X}{F}(j\omega) = \frac{k_{eq}^{-1}}{1-\left(\frac{\omega}{\omega_o}\right)^2 + j\frac{\omega}{Q\omega_o}}}$$

$$F = j\omega m_{eq}\dot{x} + \frac{k_{eq}}{j\omega}\dot{x} + c_{eq}\dot{x}$$

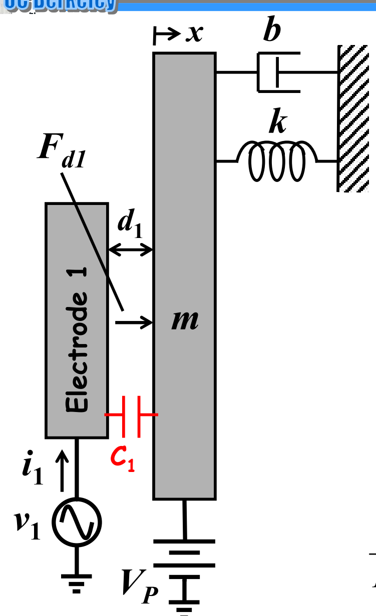
$\Rightarrow$ Converting to full phasor form:

$$F = (j\omega)(j\omega X)m_{eq} + \frac{k_{eq}}{\cancel{j\omega}}(\cancel{j\omega}X) + c_{eq}(j\omega X)$$

$$\frac{X}{F}(j\omega) = \frac{1}{k_{eq}}\left[-\omega^2\frac{m_{eq}}{k_{eq}} + 1 + j\frac{c_{eq}\omega}{k_{eq}}\right]^{-1} = \frac{1}{k_{eq}}\left[-\left(\frac{\omega}{\omega_o}\right)^2 + 1 + j\frac{\omega}{Q\omega_o}\right]^{-1}$$

$$\left[\frac{k_{eq}}{m_{eq}} = \omega_o^2,\ Q = \frac{m_{eq}\omega_o}{c_{eq}} = \frac{k_{eq}}{\omega_o c_{eq}} \rightarrow \frac{k_{eq}}{c_{eq}} = Q\omega_o\right]$$

Force-to-Velocity Relationship



- The relationship between input voltage v_1 and force F_{d1}:

$$F_{d1} \approx -V_P\frac{\partial C_1}{\partial x}v_1$$

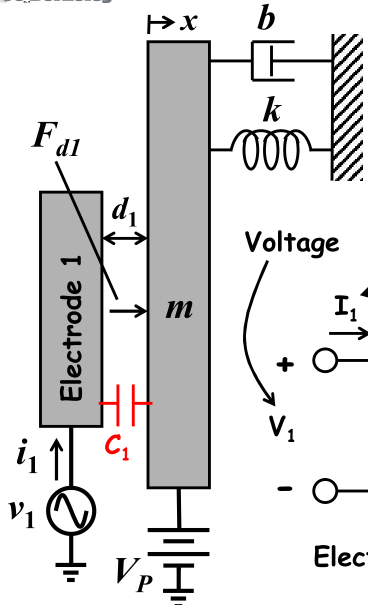
- When displacement x is the mechanical output variable:

$$\frac{X(s)}{F_{d1}(s)} = \frac{1}{k}\frac{\omega_o^2}{s^2 + (\omega_o/Q)s + \omega_o^2}$$

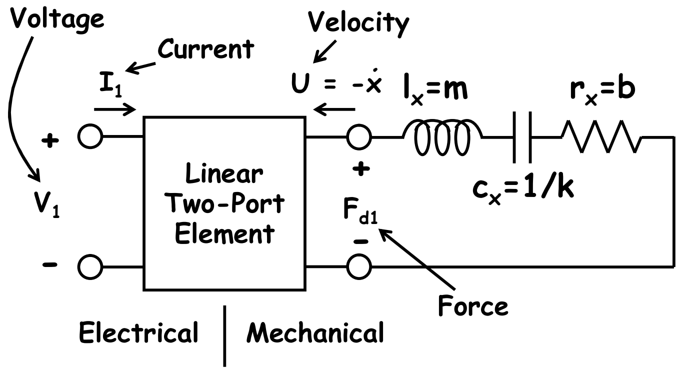
- When velocity υ is the mechanical output variable:

$$\frac{\upsilon(s)}{F_{d1}(s)} = \frac{sX(s)}{F_{d1}(s)} = \frac{1}{k}\frac{\omega_o^2 s}{s^2 + (\omega_o/Q)s + \omega_o^2}$$

Force-to-Velocity Equiv. Ckt.

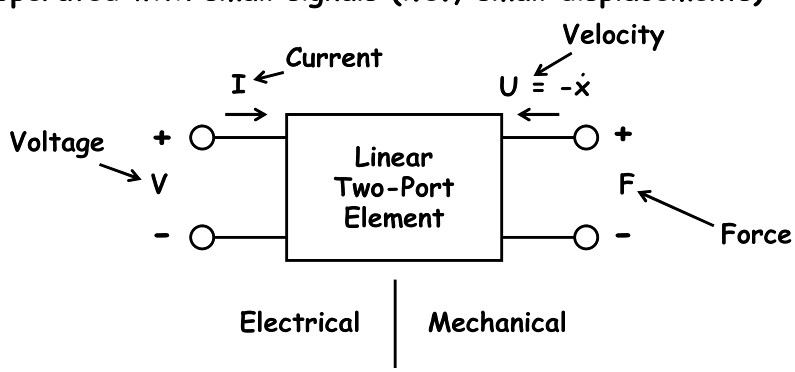


- Combine the previous lumped LCR mechanical equivalent circuit with a circuit modeling the capacitive transducer → circuit model for voltage-to-velocity

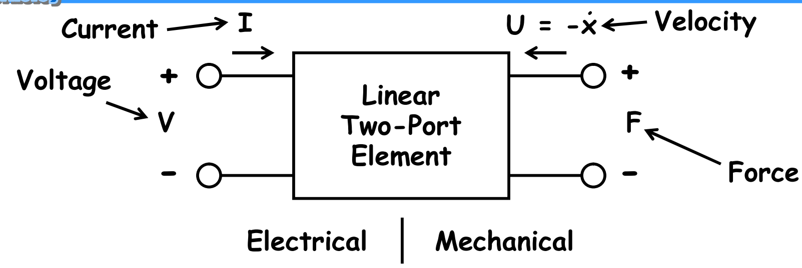


Equiv. Circuit for a Linear Transducer

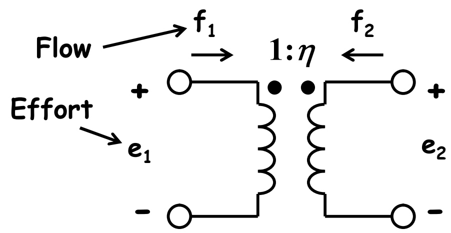
- A transducer …
  - converts energy from one domain (e.g., electrical) to another (e.g., mechanical)
  - has at least two ports
  - is not generally linear, but is virtually linear when operated with small signals (i.e., small displacements)



Equiv. Circuit for a Linear Transducer



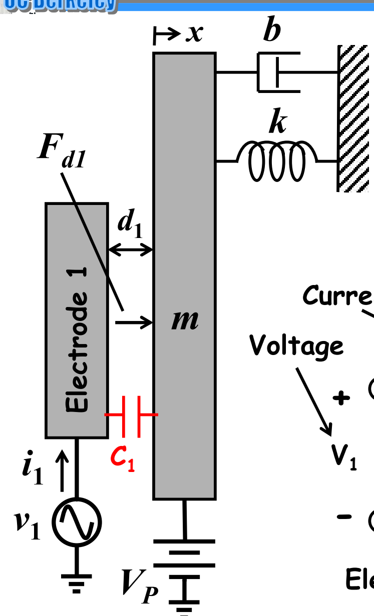
- For physical consistency, use a transformer equivalent circuit to model the energy conversion from the electrical domain to mechanical domain



$$\begin{bmatrix} e_2 \\ f_2 \end{bmatrix} = \begin{bmatrix} \eta & 0 \\ 0 & -\dfrac{1}{\eta} \end{bmatrix} \begin{bmatrix} e_1 \\ f_1 \end{bmatrix}$$

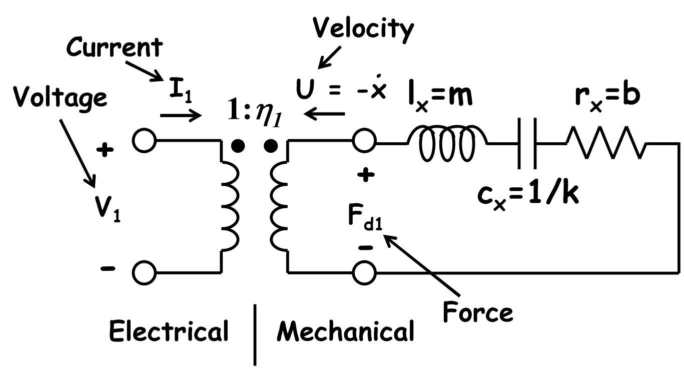
Describing Matrix

Electromechanical Equivalent Circuit



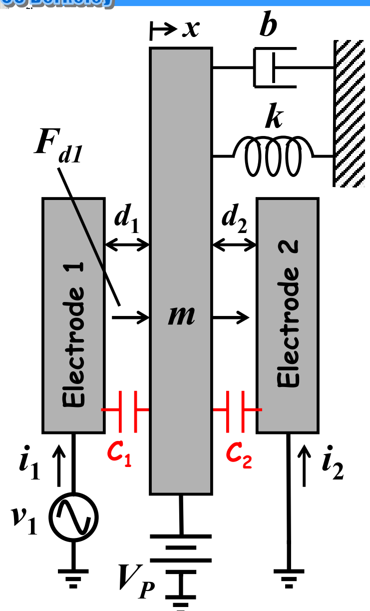
- $e_2=F_{d1}$, $e_1=v_1$, just need η_1:
- From the matrix: $e_2=\eta e_1$

$$F_{d1} \approx -V_P \frac{\partial C_1}{\partial x} v_1 \longrightarrow \boxed{\eta_1 = \left| V_P \frac{\partial C_1}{\partial x} \right|}$$



Output Modeling

Output Current Into Ground



- When the mass moves with time-dependent displacement x(t), the electrode-to-mass capacitors $C_1(x,t)$ and $C_2(x,t)$ vary with time
- This generates an output current:

$$[q = CV] \Rightarrow i = \frac{dq}{dt} = C\frac{\partial V}{\partial t} + V\frac{\partial C}{\partial t}$$

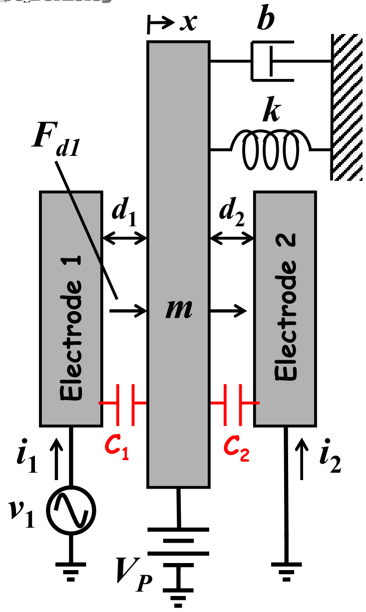
$$i_2(t) = C_2(x,t)\frac{dV_2(t)}{dt} + V_2(t)\frac{dC_2(x,t)}{dt}$$

$$[V_2(t) = -V_P] \Rightarrow i_2 = -V_P\frac{dC_2}{dt} = -V_P\frac{\partial C_2}{\partial x}\frac{\partial x}{\partial t}$$

In phasor form: $I_2(j\omega) = -V_P\frac{\partial C_2}{\partial x}(j\omega X)$

$$\boxed{I_2(j\omega) = -j\omega V_P\frac{\partial C_2}{\partial x}X}$$

Output Current Into Ground



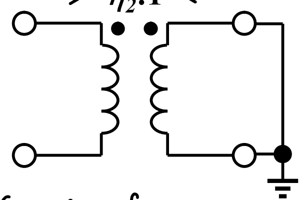
$I_2(j\omega) = \underbrace{-j}_{90^\circ \text{ phase lag}}\omega V_P \frac{\partial C_2}{\partial x} X = -V_P \frac{\partial C}{\partial x} U$

(+) (+) → $I_2 = (-)$ when $x = (+)$ ✓

- Again, model with a transformer:

Velocity → $U = \dot{x}$ Current ← I_2

$\eta_2 : 1$



$f_2 = -\frac{1}{\eta_2} f_1 \rightarrow f_1 = -\eta_2 f_2$

$[f_1 = I_2,\ f_2 = U] \Rightarrow I_2 = -\eta_2 U$

$\therefore \boxed{\eta_2 = \left| V_P \frac{\partial C}{\partial x} \right|}$

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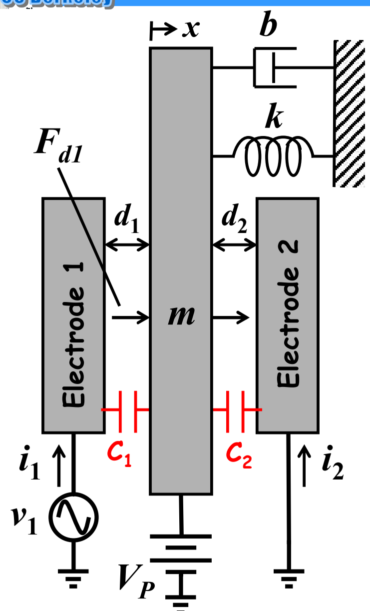
Input Current Expression



Get $I_1(j\omega)$:

$i_1(t) = C_1(x,t)\frac{dV_1(t)}{dt} + V_1(t)\frac{dC_1(x,t)}{dt}$

$[V_1(t) = v_1 - V_P] \Rightarrow i_1 = C_1 \frac{dv_1}{dt} + [v_1 - V_P]\frac{\partial C_1}{\partial x}\frac{\partial x}{\partial t}$

$\therefore I_1(j\omega) = C_1(j\omega V_1) + V_1\frac{\partial C_1}{\partial x}(j\omega X) - V_P\frac{\partial C_1}{\partial x}(j\omega X)$

$= j\omega C_1 V_1 + j\omega V_1 \frac{\partial C_1}{\partial x} X - j\omega V_P \frac{\partial C_1}{\partial x} X$

$[V_1 \ll V_P] \Rightarrow I_1(j\omega) = \underbrace{j\omega C_1 V_1}_{\text{Feedthrough Current}} - \underbrace{j\omega V_P \frac{\partial C_1}{\partial x} X}_{\text{Motional Current}}$ (due to mass motion)

@ DC: $x = \frac{F_{d1}}{k} = -\frac{1}{k}V_P\left(\frac{\partial C_1}{\partial x}\right)v_1$

@ resonance: $x = \frac{Q F_{d1}}{jk} = -\frac{Q}{jk}V_P\frac{\partial C_1}{\partial x}v_1$

90° phase lag

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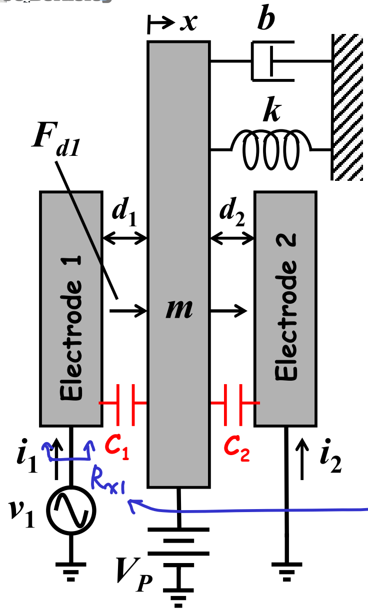
Input Current Expression (cont)



Thus: (@ resonance)

$$I_1(j\omega) = j\omega_0 C_1 V_1 + j\omega_0 \left(V_P \frac{\partial C_1}{\partial x}\right)^2 \frac{Q}{jk} V_1$$

$$= \underbrace{j\omega_0 C_1 V_1} + \underbrace{\omega_0 \frac{Q}{k}\eta^2 V_1}$$

90° phase-shifted from V_1

This is a capacitor in shunt w/ the input.

In phase w/ V_1

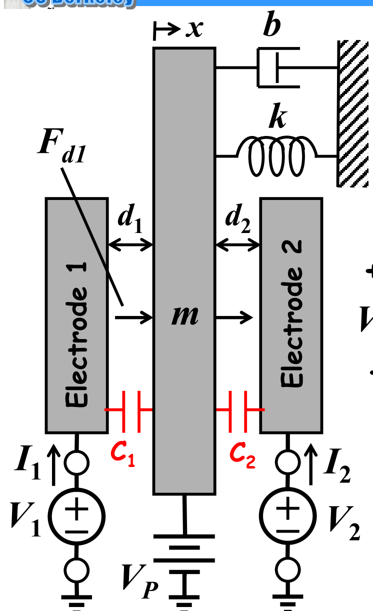
This is an effective resistance seen looking into Electrode 1

Motional Resistance:

$$R_{x1} = \frac{V_1}{I_1} = \frac{k}{\omega_0 Q \eta_1^2} = \frac{m\omega_0}{Q\eta^2} = \boxed{\frac{b}{\eta_1^2} = R_{x1}}$$

(The equivalent ckt. better get this right!)

Complete Electrical-Port Equiv. Circuit

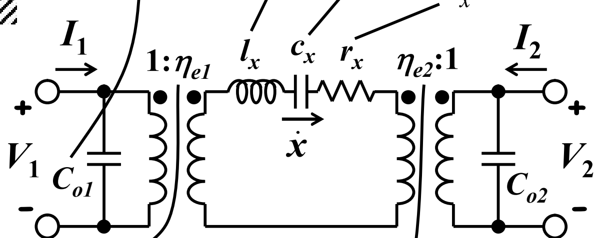


Static electrode-to-mass overlap capacitance

$l_x = m$

$c_x = \frac{1}{k}$

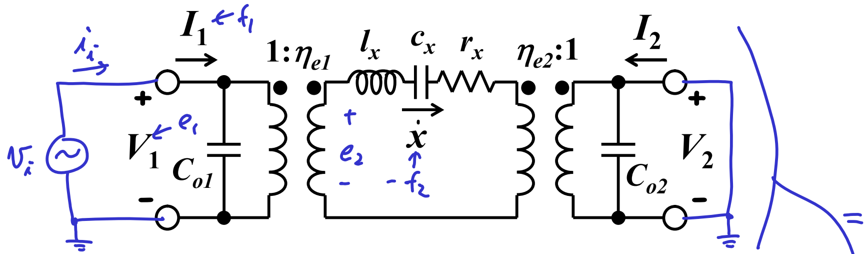
$r_x = b$



$$\eta_{e1} = V_P \frac{\partial C_1}{\partial x} = V_P \frac{C_{o1}}{d_1}$$

$$\eta_{e2} = V_P \frac{\partial C_2}{\partial x} = V_P \frac{C_{o2}}{d_2}$$

Input Impedance Into Port 1



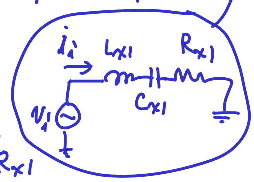
- What is the impedance seen looking into port 1 with port 2 shorted to ground?

From our transformer model: $\begin{bmatrix} e_2 \\ f_2 \end{bmatrix} = \begin{bmatrix} n & 0 \\ 0 & -\frac{1}{n} \end{bmatrix} \begin{bmatrix} e_1 \\ f_1 \end{bmatrix} \Rightarrow \begin{array}{l} e_2 = n e_1 \rightarrow e_1 = \frac{e_2}{n} \\ f_2 = -\frac{1}{n} f_1 \rightarrow f_1 = -n f_2 \end{array}$

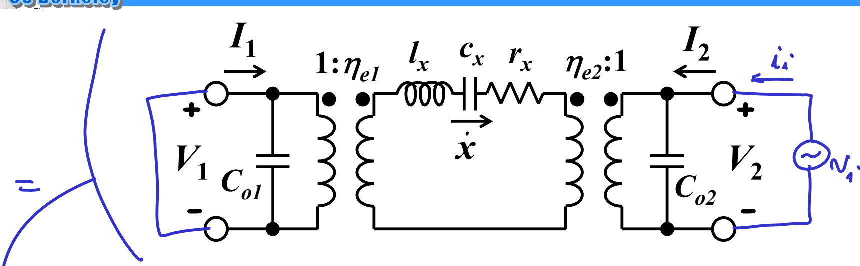
$$\frac{e_1}{f_1} = \frac{e_2}{n}\left(\frac{1}{-n f_2}\right) = -\frac{1}{n^2}\frac{e_2}{f_2} \longrightarrow \frac{v_i}{i_i} = z_i = -\frac{1}{\eta_{e1}^2}\frac{F_2}{(-\dot{x}_2)} = \frac{1}{\eta_{e1}^2} z_x$$

$$z_i = \frac{1}{\eta_{e1}^2}\left(j\omega l_x + \frac{1}{j\omega c_x} + r_x\right) = j\omega\underbrace{\left(\frac{l_x}{\eta_{e1}^2}\right)}_{L_{x1}} + \frac{1}{j\omega\underbrace{(\eta_{e1}^2 c_x)}_{C_{x1}}} + \underbrace{\frac{r_x}{\eta_{e1}^2}}_{R_{x1}}$$

=



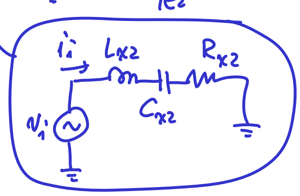
Input Impedance Into Port 2



- What is the impedance seen looking into port 2 with port 1 shorted to ground?

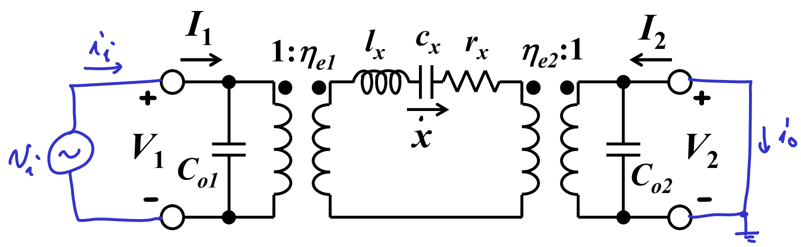
$$\frac{v_i}{i_i} = Z_i = \frac{1}{\eta_{e2}^2}\left(j\omega l_x + \frac{1}{j\omega c_x} + r_x\right) = j\omega\underbrace{\left(\frac{l_x}{\eta_{e2}^2}\right)}_{L_{x2}} + \frac{1}{j\omega\underbrace{(\eta_{e2}^2 c_x)}_{C_{x2}}} + \underbrace{\frac{r_x}{\eta_{e2}^2}}_{R_{x2}}$$

=



Note: These are not the same as L_{x1}, C_{x1}, & R_{x1}!

Port 1 to 2 TransG Across the Circuit



- What is the transconductance from port 1 to port 2 with port 2 shorted to ground? $\rightarrow \frac{i_o}{v_i}(j\omega)$

$$\dot{x} = \frac{1}{\eta_{e1}} i_i, \quad i_o = \eta_{e2}\dot{x} \Rightarrow i_o = \frac{\eta_{e2}}{\eta_{e1}} i_i = \frac{\eta_{e2}}{\eta_{e1}}\left(\frac{v_i}{z_i}\right) = \frac{\eta_{e2}}{\eta_{e1}} v_i \left[\eta_{e1}^2 \frac{1}{j\omega l_x + \frac{1}{j\omega c_x} + r_x}\right]$$

$$\therefore \frac{i_o}{v_i}(j\omega) = \frac{\eta_{e1}\eta_{e2}}{j\omega l_x + \frac{1}{j\omega c_x} + r_x} = \left[j\omega L_{x12} + \frac{1}{j\omega C_{x12}} + R_{x12}\right]^{-1} \begin{cases} L_{x12} = \frac{l_x}{\eta_{e1}\eta_{e2}} \\ C_{x12} = \eta_{e1}\eta_{e2}c_x \\ R_{x12} = \frac{r_x}{\eta_{e1}\eta_{e2}} \end{cases}$$

Port 1 to 2 v_i-to-i_o Transfer Function

$$\frac{i_o}{v_i}(j\omega) = \frac{\eta_{e1}\eta_{e2}}{j\omega l_x + \frac{1}{j\omega c_x} + r_x} = \left[j\omega L_{x12} + \frac{1}{j\omega C_{x12}} + R_{x12}\right]^{-1} \begin{cases} L_{x12} = \frac{l_x}{\eta_{e1}\eta_{e2}} \\ C_{x12} = \eta_{e1}\eta_{e2}c_x \\ R_{x12} = \frac{r_x}{\eta_{e1}\eta_{e2}} \end{cases}$$

Separate freq. response & magnitude:

$$\frac{i_o}{v_i}(s) = \frac{1}{sL_x + \frac{1}{sC_x} + R_x} = \frac{sC_x}{s^2L_xC_x + 1 + sC_xR_x} = \frac{s\left(\frac{1}{L_x}\right)}{s^2 + \frac{1}{L_xC_x} + s\left(\frac{R_x}{L_x}\right)}$$

$$\left[\frac{1}{L_xC_x} = \omega_o^2,\ Q = \frac{\omega_o L_x}{R_x} \rightarrow \frac{R_x}{L_x} = \frac{\omega_o}{Q}\right] \Rightarrow \boxed{\frac{i_o}{v_i}(s) = \underbrace{\frac{1}{R_x}}_{\text{Gain Term}} \underbrace{\frac{s(\omega_o/Q)}{s^2 + s(\omega_o/Q) + \omega_o^2}}_{\text{Bandpass Biquad}} = \frac{1}{R_x}\Theta(s)}$$

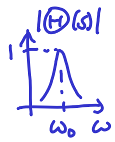
This will always be the same. Thus, could just work @ resonance & just multiply by $\Theta(s)$.

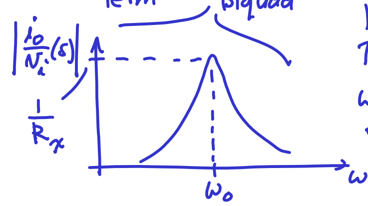
$$\Theta(s) = \frac{s(\omega_o/Q)}{s^2 + s(\omega_o/Q) + \omega_o^2}$$

$s=0$: $\Theta(0) = 0$

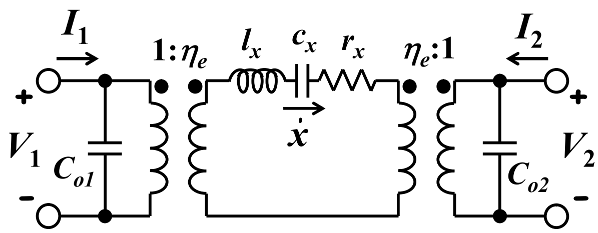
$s=j\omega_o$: $\Theta(j\omega_o) = 1$

$s=\infty$: $\Theta(\infty) = 0$



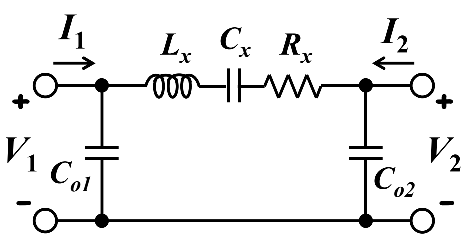


Condensed Equiv. Circuit (Symmetrical)



Holds for the symmetrical case, where port 1 and port 2 are identical

If $\eta_{e1} = \eta_{e2}$, then …



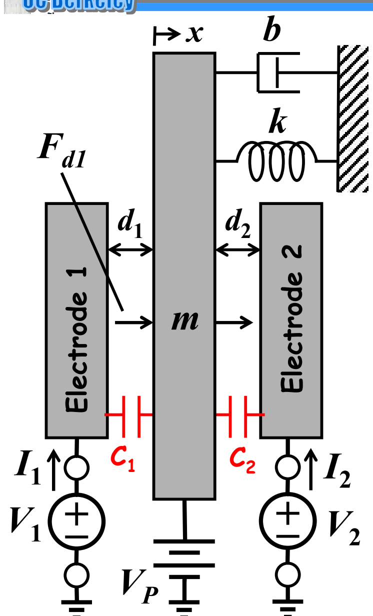
where

$$L_x = \frac{m}{\eta_e^2}$$

$$C_x = \frac{\eta_e^2}{k}$$

$$R_x = \frac{b}{\eta_e^2}$$

Phasings of Signals



- **Below**: plots of resonance electrical and mechanical signals vs. time, showing the phasings between them