Lecture Module 9: Energy Methods

Lecture Outline

- Reading: Senturia, Chpt. 10
- Lecture Topics:
  - Energy Methods
    - Virtual Work
    - Energy Formulations
    - Tapered Beam Example
Energy Methods

More General Geometries

• Euler-Bernoulli beam theory works well for simple geometries
• But how can we handle more complicated ones?
• Example: tapered cantilever beam
• Objective: Find an expression for displacement as a function of location $x$ under a point load $F$ applied at the tip of the free end of a cantilever with tapered width $W(x)$

$$W(x) = W \left( 1 - \frac{x}{2L_c} \right)$$

Top view of cantilever's $W(x)$

50% taper

$x = L_c$
Solution: Use Principle of Virtual Work

* In an energy-conserving system (i.e., elastic materials), the energy stored in a body due to the quasi-static (i.e., slow) action of surface and body forces is equal to the work done by these forces...

* Implication: if we can formulate stored energy as a function of the deformation of a mechanical object, then we can determine how an object responds to a force by determining the shape the object must take in order to minimize the difference $U$ between the stored energy and the work done by the forces:

$$U = \text{Stored Energy} - \text{Work Done}$$

* Key idea: we don't have to reach $U = 0$ to produce a very useful, approximate analytical result for load-deflection.

More Visual Description ...

Some problem as before: Take a beam & apply a force:

1. Apply force.
2. Beam responds by bending.
3. This force has done work:
   $$W = F \cdot y(x)$$
4. Strain generated → This means the beam has received an influx of stored energy
5. Then:
   $$U = \text{Stored Energy} - \text{Work Done} \rightarrow 0$$
   (We choose the right shape! This is how we get the beam's response to $F$.)
**Fundamentals: Energy Density**

- Strain energy density: \([J/m^3]\) \(W(x) = \int \sigma_y d\varepsilon_x\) density a capacitor form

To find work done in straining material

\[
W = \int_0^x \sigma_y d\varepsilon_x
\]

- Total strain energy \([J]\):

\[
W = \int_0^x E\varepsilon_x d\varepsilon_x = \frac{1}{2} E\varepsilon^2_x
\]

\[
W\{q\} = \int_0^x q d\varepsilon_x
\]

- Integrate over all strains (normal and shear)

\[
W = \iiint \left( \frac{1}{2} E\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2 + \frac{1}{2} G(\gamma_{xy}^2 + \gamma_{xz}^2 + \gamma_{yz}^2) \right) dV
\]

**Bending Energy Density**

- First, find the bending energy \(dW_{bend}\) in an infinitesimal length \(dx\):

\[
dW_{bend} = W d\varepsilon_x \int_{-h/2}^{h/2} \frac{1}{2} E\varepsilon_x(\gamma') d\gamma'
\]

\[
\varepsilon_x = \frac{d^2 y}{dx^2}
\]

\[
\frac{d^2 y}{dx^2} = \gamma' = \frac{d^2 y}{dx^2}
\]

\[
\int_{-h/2}^{h/2} \frac{1}{2} E\varepsilon_x(\gamma') d\gamma' = \frac{1}{2} E\varepsilon_x(\gamma')^2
\]

\[
dW_{bend} = W d\varepsilon_x \int_{-h/2}^{h/2} \frac{1}{2} E\varepsilon_x(\gamma')^2 d\gamma' = \frac{1}{2} E\varepsilon_x(\gamma')^2
\]

\[
W_{bend} = \frac{1}{2} E\varepsilon_x(\gamma')^2
\]
Energy Due to Axial Load

- Strain due to axial load $S$ contributes an energy $dW_{\text{stretch}}$ in length $dx$, since lengthening of the different element $dx$ (to $ds$) results in a strain $\varepsilon_x$.

$$ds = \left[ (dx)^2 + (dy)^2 \right]^{1/2} = dx \left[ 1 + \frac{1}{2} \left( \frac{dy}{dx} \right)^2 \right]^{1/2} = dx \left[ 1 + \frac{1}{2} \left( \frac{dy}{dx} \right)^2 \right]$$

$$\therefore \varepsilon_x = \frac{dx \cdot dy}{dx} = \frac{1}{2} \left( \frac{dy}{dx} \right)^2$$

Axial Strain Energy

$$dW_{\text{axial}} = S \varepsilon_x dx = S \left( \frac{1}{2} \left( \frac{dy}{dx} \right)^2 \right) dx \Rightarrow \boxed{W_{\text{axial}} = \frac{1}{2} S \int_0^L \left( \frac{dy}{dx} \right)^2 dx}$$

Shear Strain Energy

$$W_{\text{shear}} = \frac{3EI_z}{4GWh} \int_0^L \left( \frac{d^3 y}{dx^3} \right)^2 dX$$

- Shear Modulus

Applying the Principle of Virtual Work

- Basic Procedure:
  - Guess the form of the beam deflection under the applied loads
  - Vary the parameters in the beam deflection function in order to minimize:
    
    $U = \sum_j W_j - \sum_i F_i u_i$

  - Find minima by simply setting derivatives to zero

- See Senturia, pg. 244, for a general expression with distributed surface loads and body forces

Example: Tapered Cantilever Beam

- Objective: Find an expression for displacement as a function of location $x$ under a point load $F$ applied at the tip of the free end of a cantilever with tapered width $W(x)$

- Start by guessing the solution
  - It should satisfy the boundary conditions
  - The strain energy integrals shouldn’t be too tedious
  - This might not matter much these days, though, since one could just use matlab or mathematica
Strain Energy And Work By F

\[ U = W_{\text{bend}} - F \cdot y(L_c) \]

\[ W_{\text{bend}} = \frac{1}{2} E \int_0^L I_z(x) \left( \frac{d^2 y}{dx^2} \right)^2 dx \quad \text{(Bending Energy)} \]

\[ I_z(x) = \frac{W(x)h^3}{12} \quad \frac{d^2 y}{dx^2} = 2c_2 + 6c_3 x \]

\[ W(x) = W(1 - \frac{x}{2L_v}) \]

\[ \text{Tip Deflection} \]

\[ = \frac{1}{24} EWh^3 \left( 1 - \frac{x}{2L_v} \right) (2c_2 + 6c_3 x)^2 dx - F(c_2L_v^2 + c_3L_v^3) \]

Find c_2 and c_3 That Minimize U

* Minimize U \( \rightarrow \) basically, find the c_2 and c_3 that brings U closest to zero (which is what it would be if we had guessed correctly)

* The c_2 and c_3 that minimize U are the ones for which the partial derivatives of U with respective to them are zero:

\[ \frac{\partial U}{\partial c_2} = 0 \quad \frac{\partial U}{\partial c_3} = 0 \]

* Proceed:

\( \therefore \) First, evaluate the integral to get an expression for U:

\[ U = EWh^3 \left[ \frac{5c_3^2}{16} L_v^3 + \frac{c_2c_3}{3} L_v^2 + \frac{c_2^2}{8} L_v \right] - F(c_2L_v^2 + c_3L_v^3) \]
Minimize $U$ (cont)

- Evaluate the derivatives and set to zero:

\[
\frac{\partial U}{\partial c_2} = 0 = \left( \frac{EWh^3}{3} c_3 - F \right) L_e^2 + \left( \frac{EWh^3}{4} c_2 \right) L_o
\]

\[
\frac{\partial U}{\partial c_3} = 0 = \left( \frac{5}{8} EWh^3 c_3 - F \right) L_e^3 + \left( \frac{EWh^3}{3} c_2 \right) L_e^2
\]

- Solve the simultaneous equations to get $c_2$ and $c_3$:

\[
c_2 = \left( \frac{84}{13} \right) \frac{FL_e}{EWh^3}
\]

\[
c_3 = \left( \frac{24}{13} \right) \frac{F}{EWh^3}
\]

The Virtual Work-Derived Solution

- And the solution:

\[
y(x) = \left( \frac{24F}{13EWh^3} \right) \left( \frac{7}{2} \right) L_e - x^3
\]

- Solve for tip deflection and obtain the spring constant:

\[
y(L_e) = \left( \frac{24F}{13EWh^3} \right) \left( \frac{5}{2} \right) L_e^3
\]

\[
k_e = \frac{F}{y(L_e)} = \left( \frac{13EWh^3}{60L_e^3} \right)
\]

- Compare with previous solution for constant-width cantilever beam (using Euler theory):

\[
y(L_e) = \left( \frac{4F}{EWh^3} \right) L_e^3
\]

13% smaller than tapered-width case
Comparison With Finite Element Simulation

- Below: ANSYS finite element model with
  
  \[ \begin{align*}
  L &= 500 \mu m & W_{\text{base}} &= 20 \mu m & E &= 170 \text{ GPa} \\
  h &= 2 \mu m & W_{\text{tip}} &= 10 \mu m
  \end{align*} \]

- Result: (from static analysis)
  \[ k = 0.471 \mu \text{N/m} \]

- This matches the result from energy minimization to 3 significant figures

Need a Better Approximation?

- Add more terms to the polynomial
- Add other strain energy terms:
  - \( \gamma \) Shear: more significant as the beam gets shorter
  - \( \gamma \) Axial: more significant as deflections become larger
- Both of the above remedies make the math more complex, so encourage the use of math software, such as Mathematica, Matlab, or Maple
- Finite element analysis is really just energy minimization
- If this is the case, then why ever use energy minimization analytically (i.e., by hand)?
  - Analytical expressions, even approximate ones, give insight into parameter dependencies that FEA cannot
  - Can compare the importance of different terms
  - Should use in tandem with FEA for design