Normal Stress (1D)

If the force acts normal to a surface, then the stress is called a normal stress.

\[
\sigma = \frac{F}{A} \quad \text{[N/m}^2\text{; Pa]}
\]

Forces assumed uniform over the whole area A.

\[
\Rightarrow \text{Microscopic Definition: force per unit area acting on the surface of a differential volume element of a solid body.}
\]

\[
\Rightarrow \text{Note: assume shear acts uniformly across the entire surface of the element, not at just a point.}
\]

Strain (1D)

Sometimes a unit called the "microstrain" is used, where

\[
\epsilon = \frac{\Delta L}{L} \quad \text{[units]} \quad \text{[strain]} \quad \epsilon = \frac{\sigma}{E} \quad \text{[unitless]}
\]

In the elastic regime (i.e., for "small" stresses at "low" temperatures), strain is found to be proportional to stress. For solids:

\[
\epsilon = \frac{\sigma}{E} \rightarrow \quad \text{E = Young's modulus of elasticity}
\]

Thus, the units of E are the same as \( \sigma \rightarrow \text{Pa} \)

The Poisson Ratio

\[
\nu = \frac{\sigma_y}{\sigma_x} \quad \text{[unitless]}
\]

Typical values:

- Inorganic solids: 0.2 - 0.3
- Elastomers (e.g., rubber): \( \approx 0.5 \)

Shear Stress & Strain (1D)

Shear Stress:

\[
\tau = \frac{F}{A} \quad \text{[Pa]}
\]

Note: Assume compensating forces are applied to the vertical faces to avoid a net torque. (This by convention.)

Shear Strain:

\[
\gamma = \frac{\tau}{G} \quad \text{[shear modulus]}
\]

\[
G = \frac{E}{2(1+\nu)}
\]
### 2D and 3D Considerations

**Important assumption:** the differential volume element is in static equilibrium → no net forces or torques (i.e., rotational movements)

- Every \( \sigma \) must have an equal \( \sigma \) in the opposite direction on the other side of the element
- For no net torque, the shear forces on different faces must also be matched as follows:

\[
\tau_{xy} = \tau_{yx} \\
\tau_{xz} = \tau_{zx} \\
\tau_{yz} = \tau_{zy}
\]

---

### 2D Strain

**In general, motion consists of:**
- Rigid-body displacement (motion of the center of mass)
- Rigid-body rotation (rotation about the center of mass)
- Deformation relative to displacement and rotation

**Must work with displacement vectors**

**Differential definition of axial strain:**

\[
\varepsilon_x = \frac{u_x(x + \Delta x) - u_x(x)}{\Delta x} = \frac{\partial u_x}{\partial x}
\]

---

### 2D Shear Strain

**Volume Change for a Uniaxial Stress**

Given an \( x \)-directed uniaxial stress, \( \sigma_x \):

\[
\Delta x \rightarrow \Delta x (1 + \varepsilon_x) \\
\Delta y \rightarrow \Delta y (1 - \gamma_{xy}) \\
\Delta z \rightarrow \Delta z (1 - \gamma_{xz})
\]

The resulting change in volume \( \Delta V \):

\[
\Delta V = \Delta x \Delta y \Delta z (1 + \varepsilon_x)(1 - \gamma_{xy})(1 - \gamma_{xz})
\]

For small strains,

\[
\Delta V \approx \Delta x \Delta y \Delta z (1 + \varepsilon_x)(1 - \gamma_{xy})(1 - \gamma_{xz}) - \Delta x \Delta y \Delta z
\]

For \( \gamma = 0.5 \) (rubber) → no \( \Delta V \)!

For \( \gamma < 0.5 \) → finite \( \Delta V \)
Isotropic Elasticity in 3D

- Isotropic = same in all directions
- The complete stress-strain relations for an isotropic elastic solid in 3D: (i.e., a generalized Hooke’s Law)

\[
\begin{align*}
\varepsilon_x &= \frac{1}{E} \left[ \sigma_x - \nu (\sigma_y + \sigma_z) \right] \\
\gamma_{xy} &= \frac{1}{G} \tau_{xy} \\
\varepsilon_y &= \frac{1}{E} \left[ \sigma_y - \nu (\sigma_x + \sigma_z) \right] \\
\gamma_{yz} &= \frac{1}{G} \tau_{yz} \\
\varepsilon_z &= \frac{1}{E} \left[ \sigma_z - \nu (\sigma_x + \sigma_y) \right] \\
\gamma_{zx} &= \frac{1}{G} \tau_{zx}
\end{align*}
\]

Basically, add in off-axis strains from normal stresses in other directions.

Important Case: Plane Stress

- Common case: very thin film coating a thin, relatively rigid substrate (e.g., a silicon wafer)

Thin film \[\rightarrow\] Plane stress region \[\rightarrow\] Edge region

- At regions more than 3 thicknesses from edges, the top surface is stress-free \(\rightarrow \sigma_z = 0\)
- Get two components of in-plane stress:

\[
\begin{align*}
\varepsilon_x &= \frac{1}{E} [\sigma_x - \nu (\sigma_y + 0)] \\
\varepsilon_y &= \frac{1}{E} [\sigma_y - \nu (\sigma_x + 0)]
\end{align*}
\]

Important Case: Plane Stress (cont.)

- Symmetry in the xy-plane \(\rightarrow \sigma_x = \sigma_y = \sigma\)
- Thus, the in-plane strain components are: \(\varepsilon_x = \varepsilon_y = \varepsilon\) where

\[
\varepsilon_x = \left(\frac{1}{E}\right) [\sigma - \nu \sigma] = \frac{\sigma}{E/(1-\nu)} = \frac{\sigma}{E'}
\]

and where

Biaxial Modulus \(E' = \frac{E}{1-\nu}\)

Edge Region of a Tensile (\(\sigma > 0\)) Film

Net non-zero in-plane force (that we just analyzed)

At free edge, in-plane force must be zero

Film must be bent back, here

There’s no Poisson contraction, so the film is slightly thicker, here

Discontinuity of stress at the attached corner \(\rightarrow\) stress concentration

Peel forces that can peel the film off the surface
Linear Thermal Expansion

- As temperature increases, most solids expand in volume
- Definition: linear thermal expansion coefficient

\[ \alpha_T = \frac{d \varepsilon_x}{dT} \text{ [Kelvin}^{-1}] \]

**Remarks:**
- \( \alpha_T \) values tend to be in the 10^{-6} to 10^{-7} range
- Can capture the 10^{-6} by using dimensions of \( \mu \text{strain/K} \), where 10^{-6} K^{-1} = 1 \( \mu \text{strain/K} \)
- In 3D, get volume thermal expansion coefficient
- For moderate temperature excursions, \( \alpha_T \) can be treated as a constant of the material, but in actuality, it is a function of temperature

\[ \Delta V/V = 3 \alpha_T \Delta T \]

\[ \Delta = \Delta T = \text{thermal expansion} \]

Thin-Film Thermal Stress

- Assume film is deposited stress-free at a temperature \( T_r \), then the whole thing is cooled to room temperature \( T_r \)
- Substrate much thicker than thin film → substrate dictates the amount of contraction for both it and the thin film

**Remarks:**
- Cubic symmetry implies that \( \alpha \) is independent of direction

Linear Thermal Expansion as a Function of Temperature

**Remarks:**
- Cubic symmetry implies that \( \alpha \) is independent of direction

**Example:**
- Thin film: \( a_T = 7 \times 10^{-6} \text{ K}^{-1} \)
- Substrate: \( a_{TS} = 2.8 \times 10^{-6} \text{ K}^{-1} \)
- Film thickness is 10\( \mu \)m

\[ \varepsilon_{film, \text{free}} = -a_{TS} \Delta T \]

\[ \varepsilon_{film, \text{attached}} = -a_{TS} \Delta T \]

**Example:**
- Polymer film: \( a_T = 7 \times 10^{-6} \text{ K}^{-1} \)
- Metal film: \( a_T = 2 \times 10^{-6} \text{ K}^{-1} \)
- Substrate: \( a_{TS} = 2.8 \times 10^{-6} \text{ K}^{-1} \)
- Film thickness: 10\( \mu \)m

\[ \varepsilon_{film, \text{free}} = -a_{TS} \Delta T \]

**Calculation:**
- Thermal mismatch strain
- Biaxial stress

**Note:**
- Biaxial stress can only be developed by an in-plane biaxial stress.
MEMS Material Properties

Young's Modulus Versus Density

Material Properties for MEMS

<table>
<thead>
<tr>
<th>Material</th>
<th>Density, $\rho$, Kgm$^{-3}$</th>
<th>Modulus, $E$, GPa</th>
<th>$E/\rho$ GNa/kg.m$^{-1}$</th>
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</thead>
<tbody>
<tr>
<td>Silicon</td>
<td>2330</td>
<td>165</td>
<td>72</td>
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<tr>
<td>Silicon Oxide</td>
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<td>Diamond</td>
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<td>1035</td>
<td>295</td>
</tr>
</tbody>
</table>

$\sqrt{(E/\rho)}$ is acoustic velocity [

Lines of constant acoustic velocity

[Mark Spearing, MIT]

[Ashby, Mechanics of Materials, Pergamon, 1992]