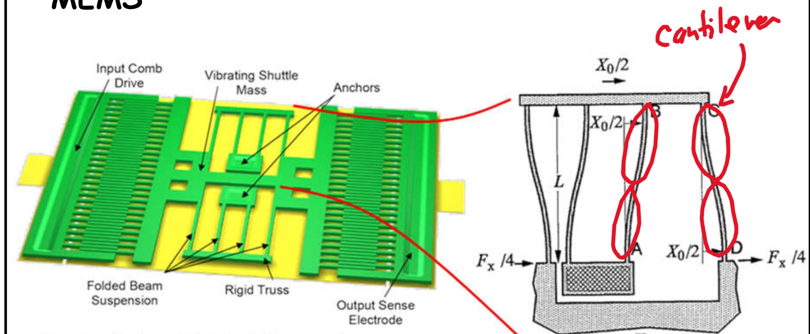
Lecture 13: Beam Bending

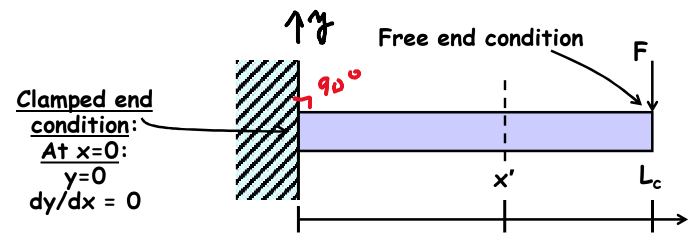
- Announcements:
- HW#3 due Thursday, 3/7, at 9 a.m.
- Module 8 on "Microstructural Elements" online
- HW#4 online and due Tuesday, 3/19, 9 a.m.
- Midterm less than 3 weeks away
- ----------------------------------
- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients
- ----------------------------------
- Last Time:
- Looking at forces & moments in equilibrium
- Now, continue with this …

- Springs and suspensions very common in MEMS
- Coils are popular in the macro-world; but not easy to make in the micro-world
- Beams: simpler to fabricate and analyze; become "stronger" on the micro-scale → use beams for MEMS

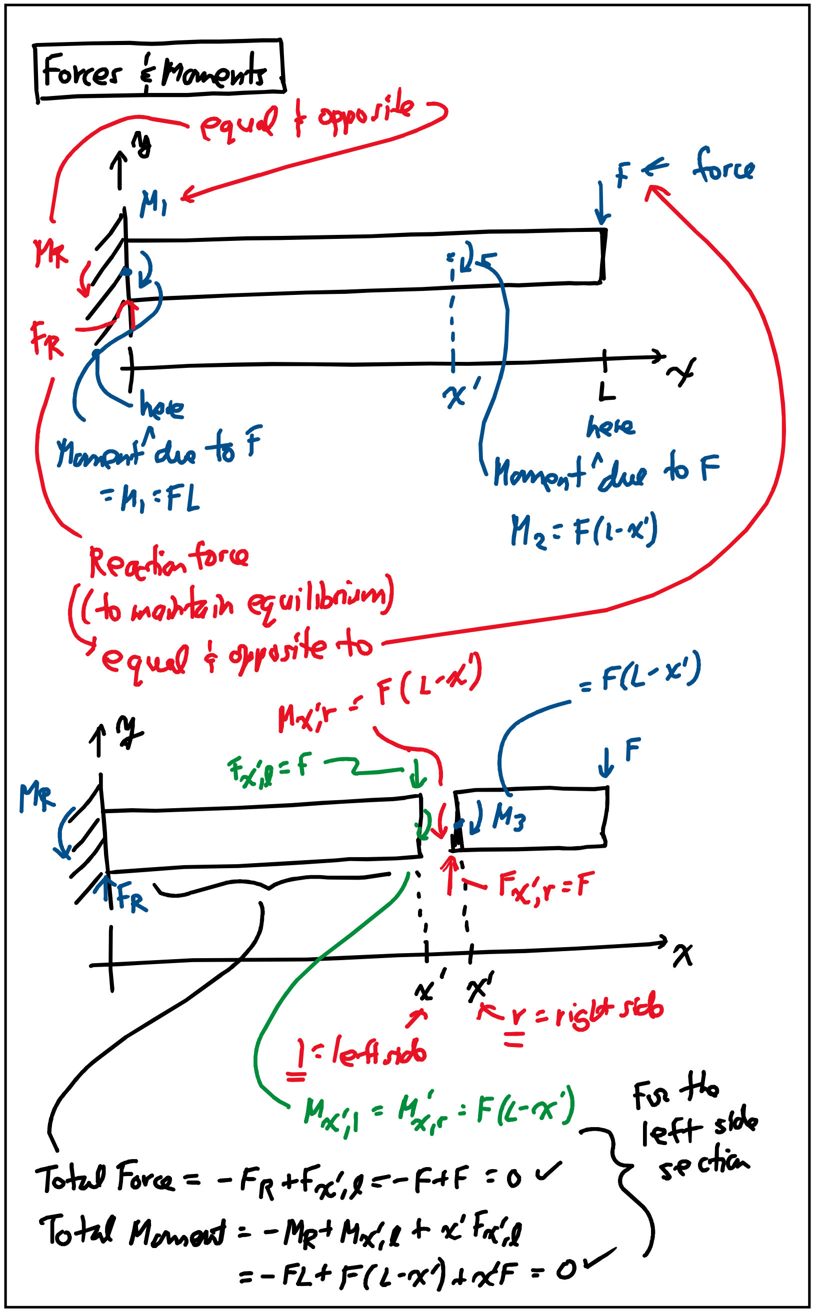


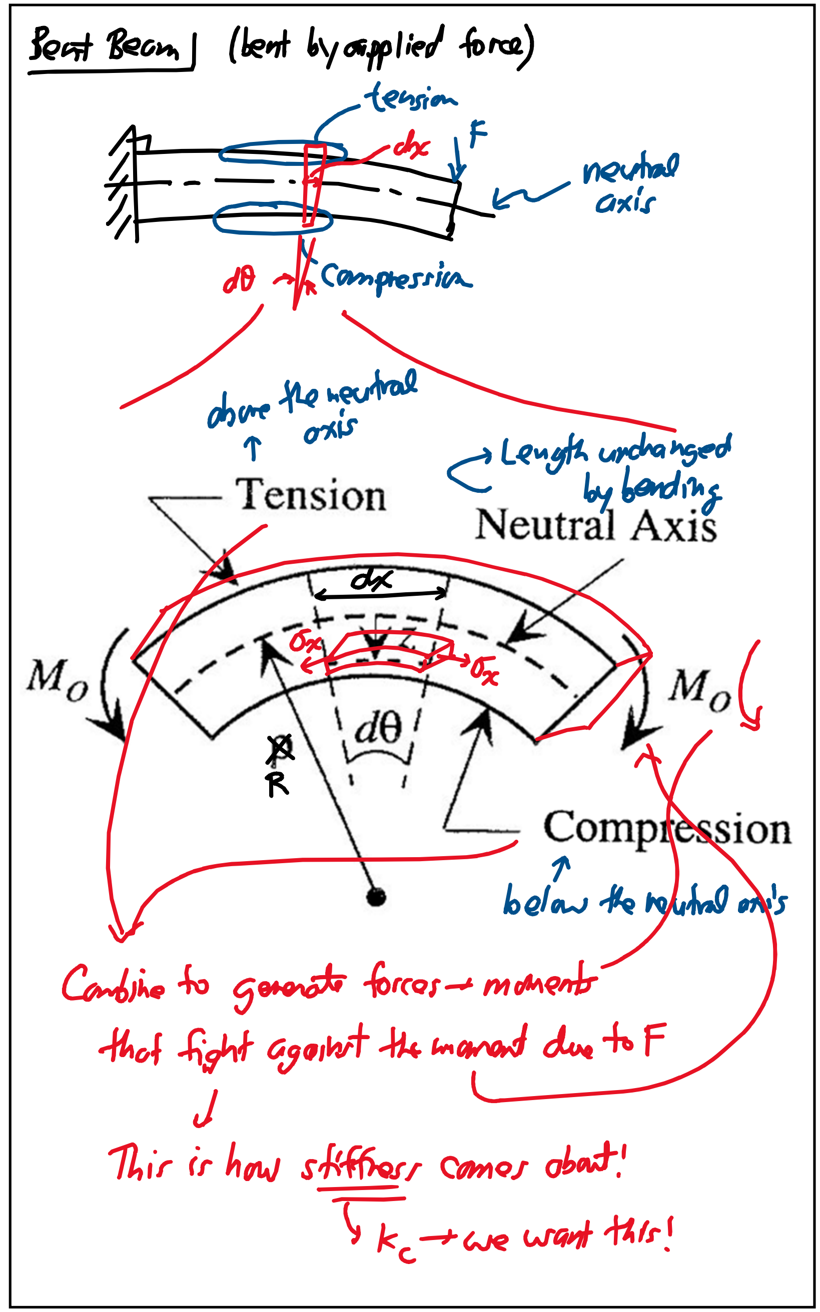
Comb-Driven Folded Beam Actuator

Problem: Bending a Cantilever Beam



- Objective: Find relation between tip deflection $y(x=L_c)$ and applied load F
- Assumptions:
  1. Tip deflection is small compared with beam length
  2. Plane sections (normal to beam's axis) remain plane and normal during bending, i.e., "pure bending"
  3. Shear stresses are negligible


Forces & Moments
equal & opposite
y
M1
F
force
MR
FR
x'
L
x
here
Moment due to F
= M1 = FL
here
Moment due to F
M2 = F(L-x')
Reaction force
((to maintain equilibrium)
equal & opposite to
Mx',r = F(L-x')
= F(L-x')
Fx',l = F
F
M3
Fx',r = F
l = left side
r = right side
Mx',l = Mx',r = F(L-x')
For the left side section
Total Force = -FR + Fx',l = -F + F = 0 ✓
Total Moment = -MR + Mx',l + x' Fx',l
= -FL + F(L-x') + x'F = 0 ✓


Bent Beam (bent by applied force)
tension
dx
F
neutral axis
dθ
compression
above the neutral axis
Tension
Length unchanged by bending
Neutral Axis
dx
σx
σx
Mo
Mo
dθ
R
Compression
below the neutral axis
Combine to generate forces → moments
that fight against the moment due to F
This is how stiffness comes about!
kc → we want this!

Beam Segment in Pure Bending

⇒ consider the segment bounded by the dashed lines defining $d\theta$

At $z=0$: neutral axis → segment length $= dx = R\,d\theta$ (1)

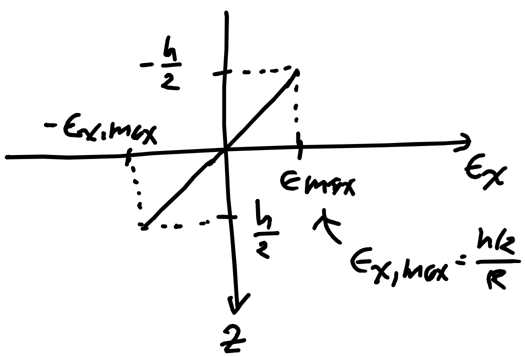
At any z: segment length $= dL = (R-z)\,d\theta$ (2)

Combine (1) & (2): $dL = dx - z\,d\theta = dx - \frac{z}{R}dx$

Thus, the axial strain @ z:

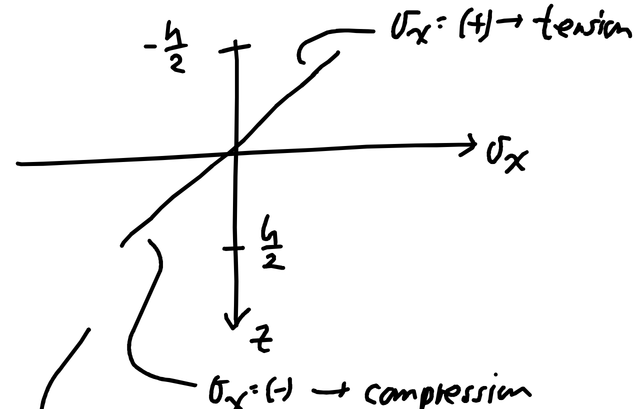
$$\epsilon_x = \frac{dL - dx}{dx} = \boxed{-\frac{z}{R} = \epsilon_x}$$

Thus, the strain varies linearly along the beam thickness:



Of course, there is a corresponding axial stress:

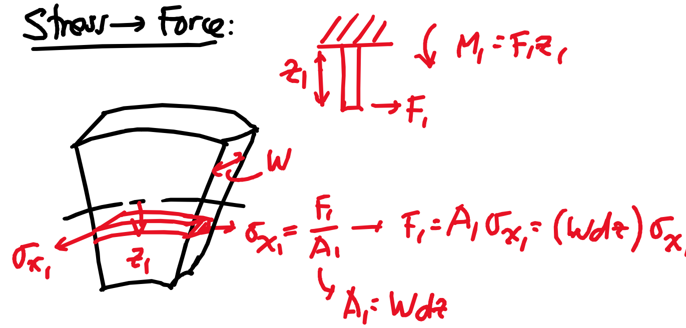
$$\sigma_x = \epsilon_x E = \boxed{-\frac{zE}{R} = \sigma_x}$$



This gradient of stress generates a bending moment!

In response to the original applied moment (generated by F)

Stress → Force:



⇒ Integrate stress through the thickness of the beam:

$$M = \int_{-\frac{h}{2}}^{\frac{h}{2}} \underbrace{[(W dz)\sigma_x]}_{\text{force}} \cdot z$$

$$= \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{EWz^2}{R} dz \Rightarrow \boxed{M = -\left(\frac{1}{12}Wh^3\right)\frac{E}{R}}$$

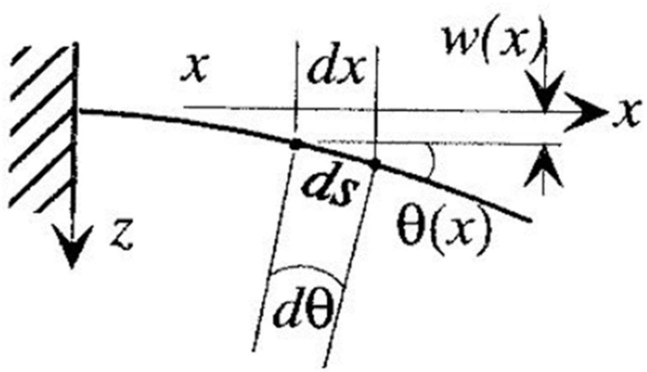
$$\left[\sigma_x = -\frac{zE}{R}\right]$$

$\frac{1}{12}Wh^3 = I \triangleq$ Moment of Inertia

$$\boxed{\frac{1}{R} = -\frac{M}{EI}}$$

Note: (+) radius of curvature ↙ (–) internal bending moment

Differential Equation for Beam Bending



Write out some geometric relationships:

⇒ then use small angle approx:

$$\cos\theta = \frac{dx}{ds} \rightarrow ds = \frac{dx}{\cos\theta} \longrightarrow ds \cong dx$$

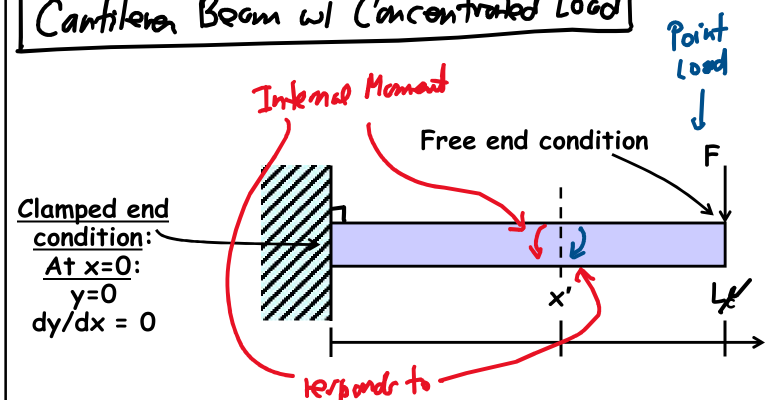
$$\tan\theta = \frac{dw}{dx} = \text{slope of the beam @ this point} \longrightarrow \theta \approx \frac{dw}{dx} \quad (1)$$

$$ds = R\,d\theta \rightarrow \frac{1}{R} = \frac{d\theta}{ds} \longrightarrow \frac{1}{R} = \frac{d\theta}{dx} \quad (2)$$

Insert (1) into (2):

$$\frac{1}{R} = \boxed{\frac{d^2w}{dx^2} = -\frac{M}{EI}}$$ ← Diff. Eqn. for Small Angle Beam Bending

Cantilever Beam w/ Concentrated Load



Internal Moment @ position x: $M = -F(L-x)$

Thus: $\frac{d^2w}{dx^2} = \frac{F}{EI}(L-x)$

w/ Clamped End B.C.: $w(x=0)=0$, $\frac{dw}{dx}(x=0)=0$
Free-End B.C: none

Solve to get w:

⇒ use Laplace; or a trial solution:

$w = A + Bx + Cx^2 + Dx^3$, then apply B.C.'s

$$w = \frac{FL}{2EI}x^2\left(1-\frac{x}{3L}\right)$$

↑ Deflection @ x due to a point load F applied @ $x=L$

Maximum Deflection → occurs @ $x=L$:

$$w_{max} = \left(\frac{L^3}{3EI}\right)F \rightarrow F = \left(\frac{3EI}{L^3}\right)w(x=L) = k_c w(x=L)$$

stiffness $\triangleq k_c$ @ $x=L$

where $k_c = \frac{3EI}{L^3}$

$\left[I = \frac{1}{12}Wh^3\right]$ → $k_c = \frac{1}{4}EW\frac{h^3}{L^3}$

Can now be used to solve general beam systems!

Ex. $L = 100\mu m$, $W = 2\mu m$, $h = 2\mu m$

polysilicon → $E = 150GPa$

$$k_c = \frac{1}{4}(150G)(2\mu)\left(\frac{2\mu}{100\mu}\right)^3 = \underline{0.6 N/m}$$

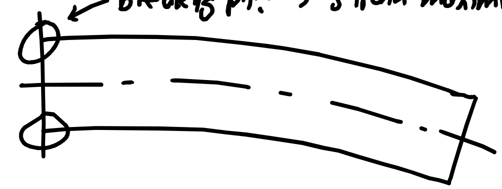
Maximum Stress in a Bent Cantilever

From before, the radius of curvature is given by

$$\frac{1}{R} = \frac{d^2w}{dx^2} = \frac{F}{EI}(L-x)$$

⇒ $\frac{1}{R}$ maximizes (i.e., R minimizes) when

$$x=0: \quad \frac{1}{R} = \frac{d^2w}{dx^2} = \frac{FL}{EI}$$



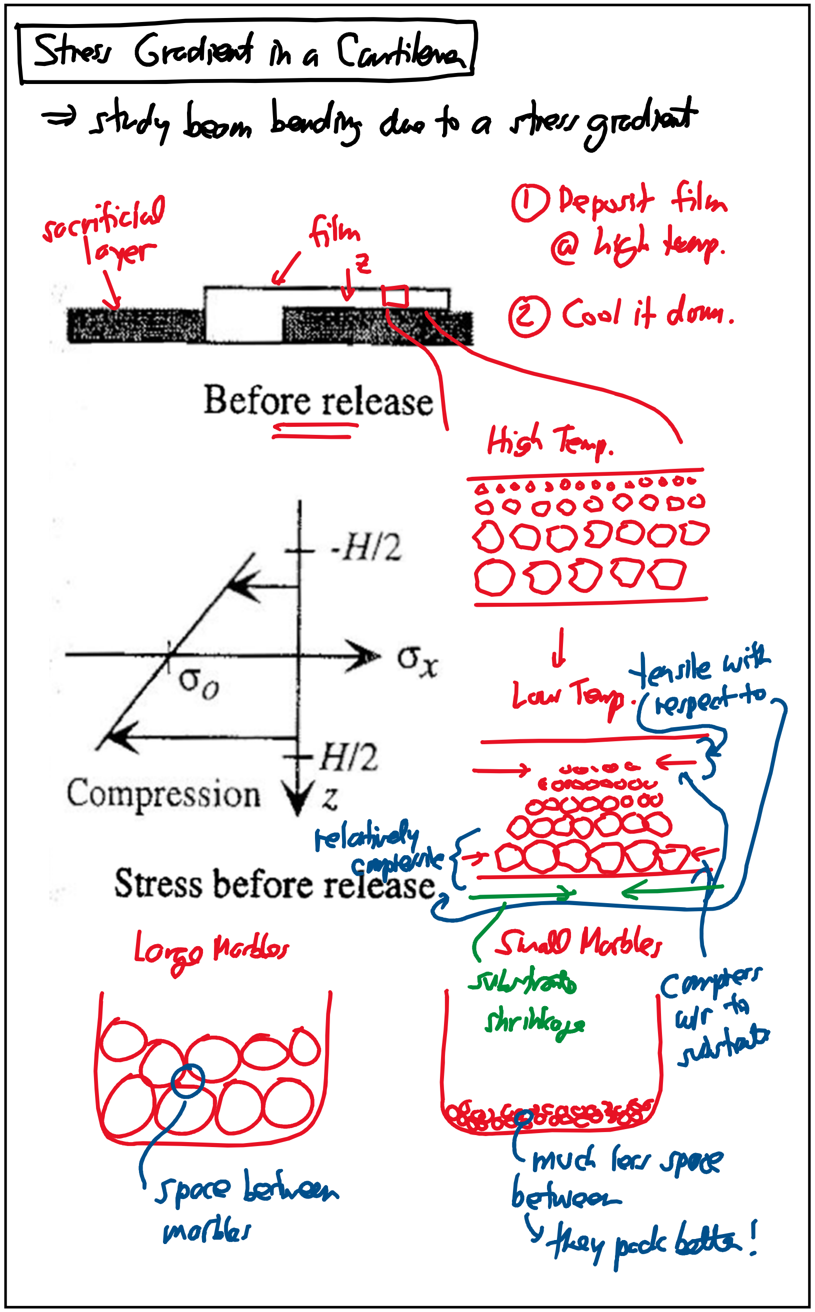
Strain maximizes:

① At top surface → tensile

② At bottom surface → compressive

$$\epsilon_{max} = \frac{z}{R} = \frac{h}{2}\frac{1}{R} = \frac{h}{2}\frac{FL}{EI} = \epsilon_{max}$$

$\left[I = \frac{1}{12}Wh^3\right]$ ⇒ $$\epsilon_{max} = \frac{6L}{EWh^2}F$$


Stress Gradient in a Cantilever
⇒ study beam bending due to a stress gradient
sacrificial layer
film
z
① Deposit film @ high temp.
② Cool it down.
Before release
High Temp.
Low Temp.
tensile with respect to
-H/2
σ_x
σ_o
H/2
Compression
z
relatively compressive
Stress before release
Large Marbles
Small Marbles
substrate shrinkage
compress w/r to substrate
space between marbles
much less space between
they pack better!

