**Lecture 15: Beam Combos II**

- Announcements:
- HW#4 online, due Tuesday, 3/19, 9 a.m.
- Midterm Exam: Thursday, March 21, 11-12:30 p.m., 293 Cory (right here)
- No lecture next Tuesday, 3/19
  - The EECS Faculty Retreat is this day
  - I will post a video lecture instead
- -----------------------------------
- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients
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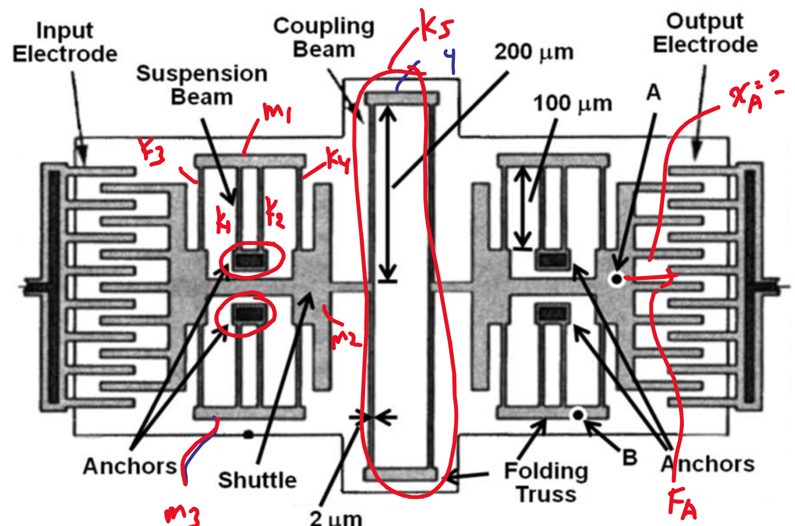
- Last Time:
- Spring circuits
- Continue with this

over

**Analyzing an Interconnected Ensemble of Beams (Springs) & Masses**




Typical Questions: → all demand knowledge of $x = f(F)$

① How does the structure move in response to a force at a specific location. ← in turn requires that we know stiffness!

② What is the frequency response to an AC force applied at a specific location.

③ Noise?

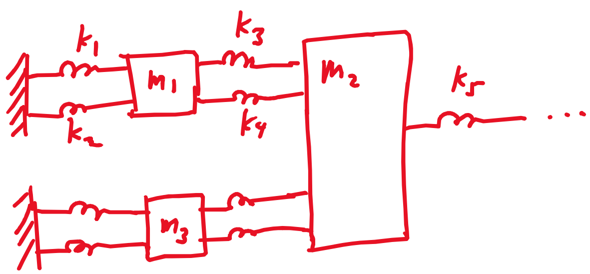
④ Response to environmental stimuli? (e.g., rotation)

⑤ How does stress affect the behavior of the structure?

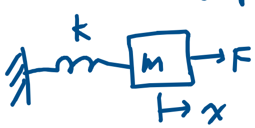
**Procedure:**

① Build the ckt. (extract the ckt.)
↳ in the x-direction (for this example)

Anchor



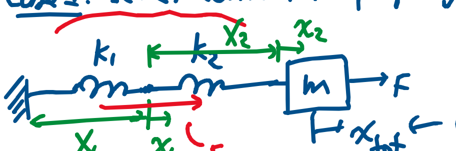

② Analyze to get $x = f(F)$ ← force
↑ displacement




$\Rightarrow F = kx \rightarrow x = \frac{F}{k} = \left(\frac{1}{k}\right)F$
↑ compliance

series because one must go thru both $k_1$ & $k_2$ to get from anchor to forcing pt.

(a) Case 1: series connection of springs




$x_1$ ↑ original spring length

$x_1$ ↑ Δx increase as response to force

F ← thru variable

$x_{tot}$ ← want this ∴ need $k_{tot}$

$x_{tot} = \left(\frac{1}{k_{tot}}\right)F$

$x_{tot} = x_1 + x_2 = F\left(\frac{1}{k_1} + \frac{1}{k_2}\right) = \frac{F}{(k_1 \| k_2)} = \frac{F}{k_{tot}}$

$\left[x_1 = \frac{F}{k_1}, x_2 = \frac{F}{k_2}\right]$

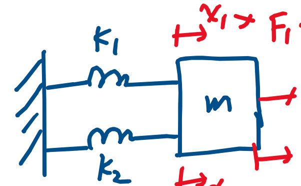
$\left[\text{"}\|\text{" operator} \triangleq A \| B = \frac{1}{\frac{1}{A} + \frac{1}{B}} = \frac{AB}{A+B}\right]$

$\boxed{k_{tot} = k_1 \| k_2}$ (for $k_1$ & $k_2$ in series)

For EE's:


↳ springs combine like capacitors

(b) Case 2: parallel springs — parallel if can go from anchor to forcing pt. via only one of the springs



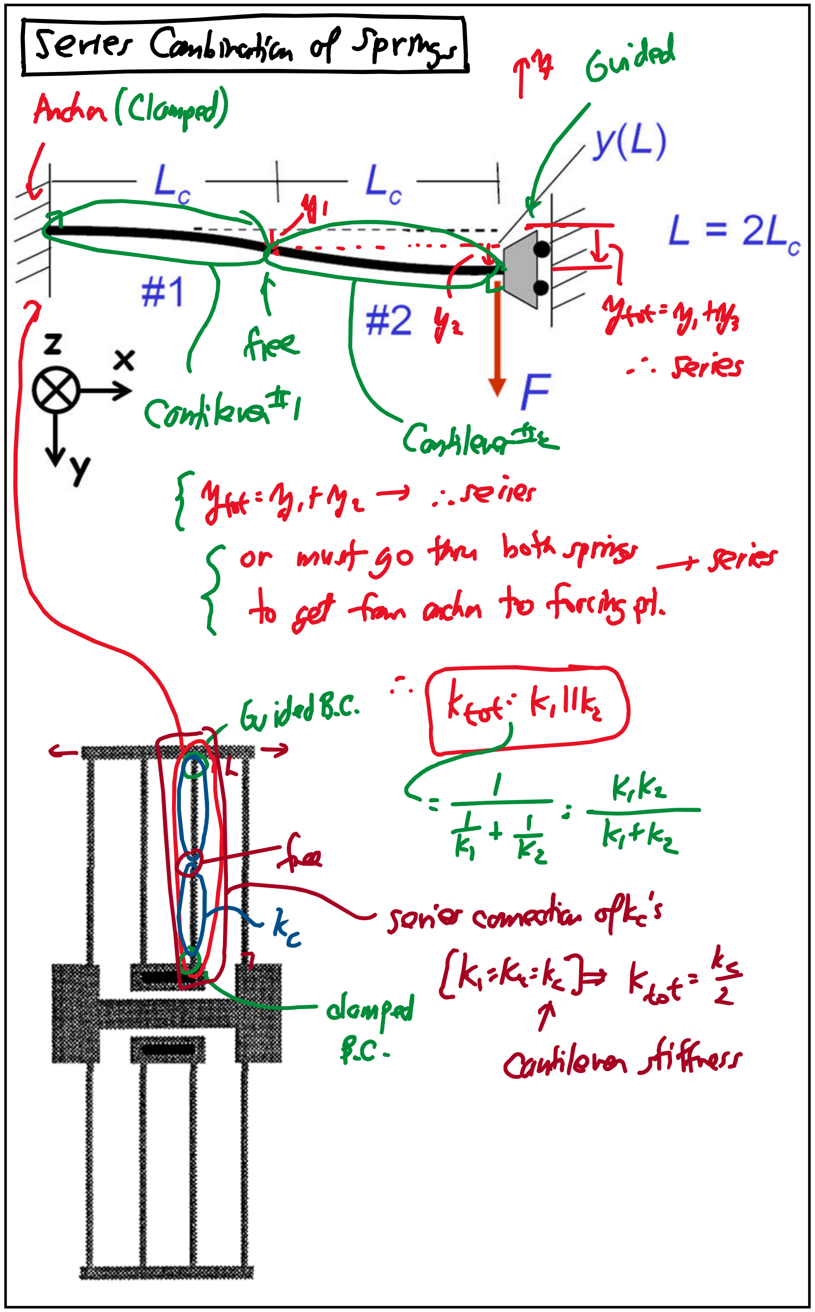

$F_1 = k_1 x_{tot}$

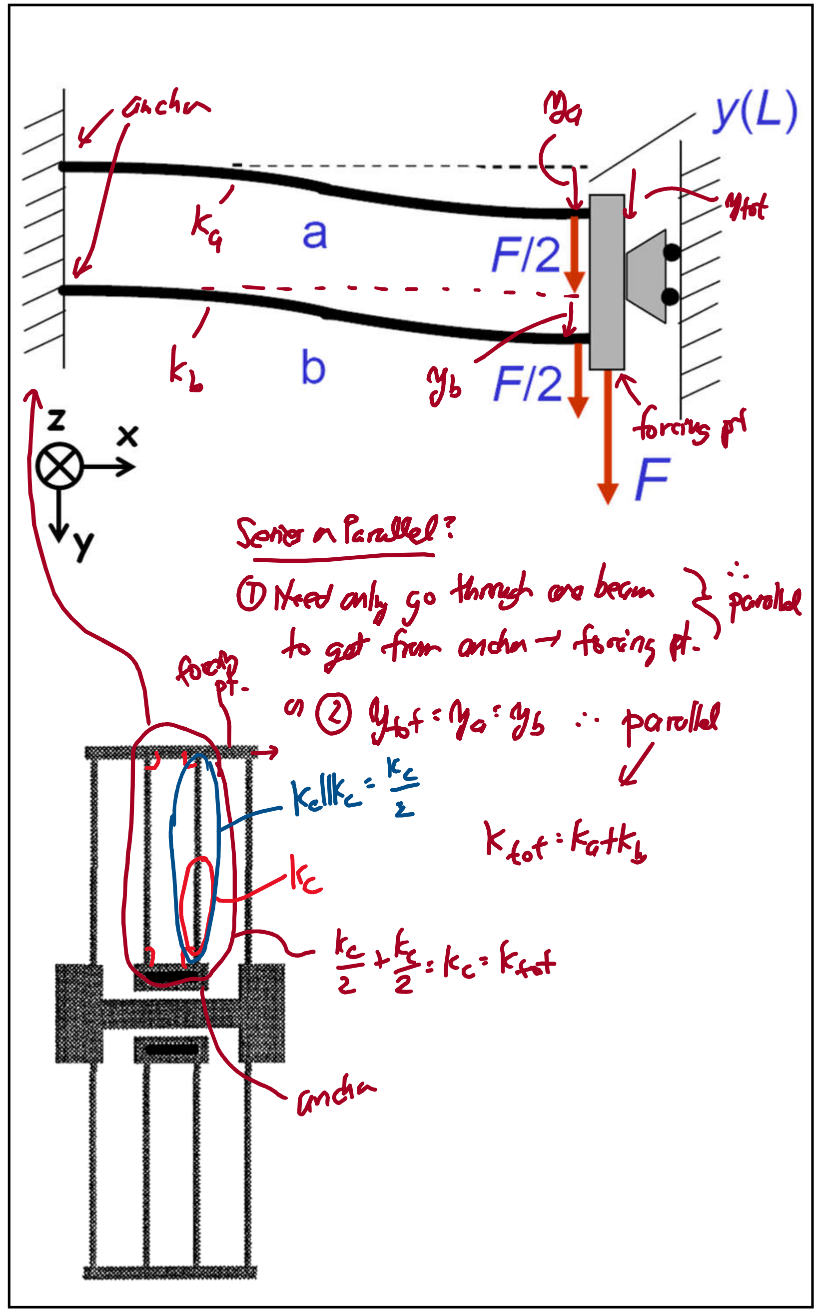
$F_2 = k_2 x_{tot}$

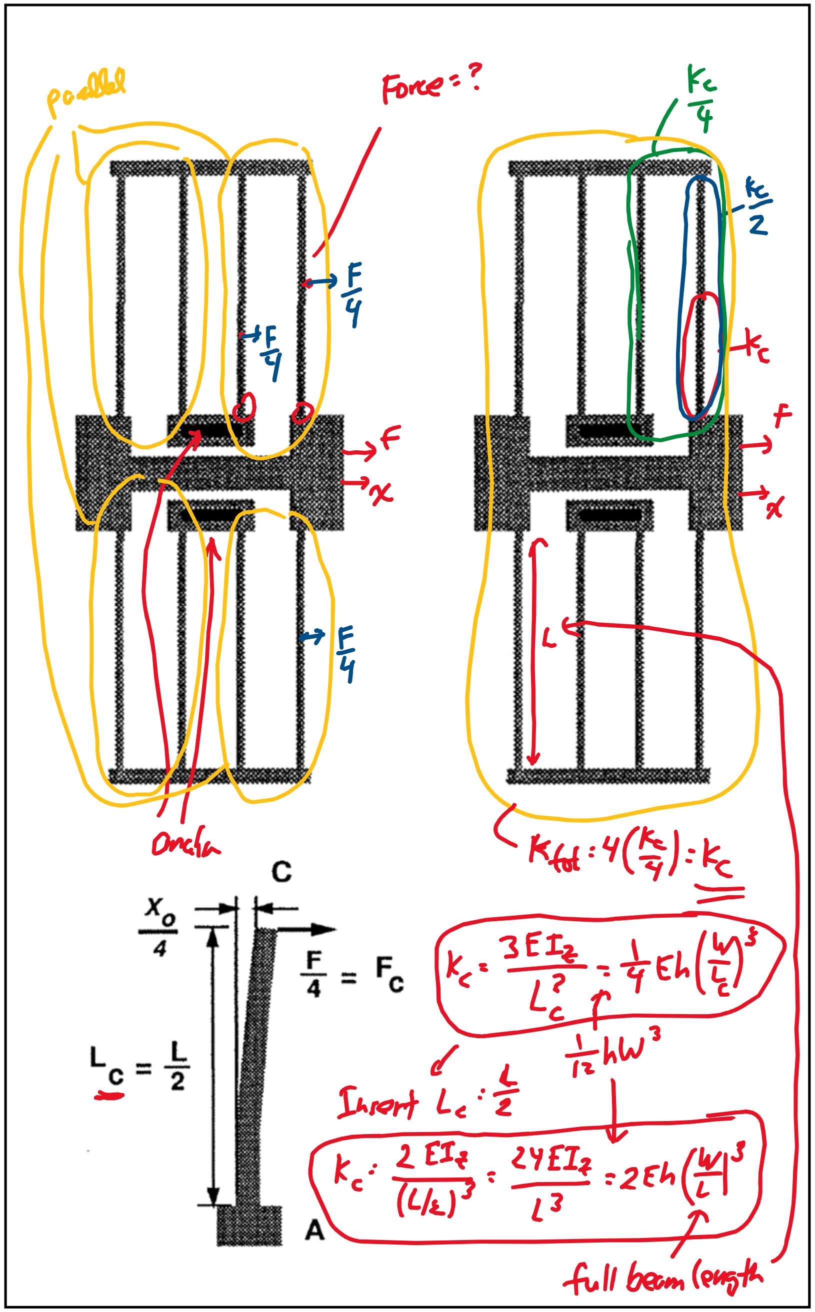
$x_{tot} = x_1 = x_2$ — another way to tell that springs are in parallel!

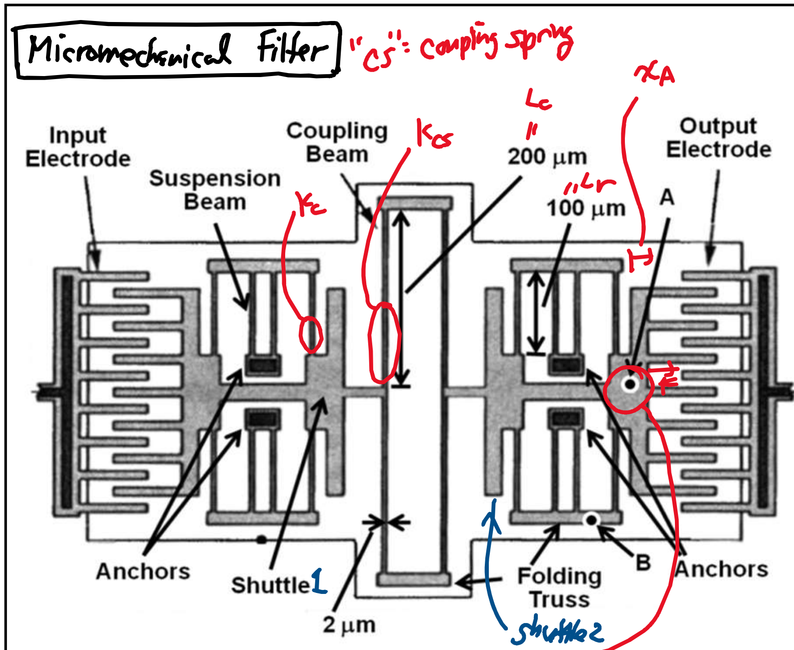
$F = F_1 + F_2 = (k_1 + k_2)x_{tot}$
($k_1 + k_2$ = $k_{tot}$)

$\boxed{k_{tot} = k_1 + k_2}$ (for $k_1$ & $k_2$ in parallel)


Series Combination of Springs
Anchor (Clamped)
Guided
y(L)
$L_c$
$L_c$
$L = 2L_c$
#1
#2
F
z
x
y
free
Cantilever #1
Cantilever #2
$y_{tot} = y_1 + y_2$ → ∴ series
or must go thru both springs to get from anchor to forcing pt. → series
Guided B.C.
∴ $k_{tot} = k_1 \| k_2$
$= \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}} = \frac{k_1 k_2}{k_1 + k_2}$
free
$k_c$
series connection of $k_c$'s
clamped B.C.
$[k_1 = k_2 = k_c] \Rightarrow k_{tot} = \frac{k_c}{2}$
cantilever stiffness


anchor
$y_a$
y(L)
$y_{tot}$
$k_a$
a
F/2
$k_b$
b
$y_b$
F/2
forcing pt
F
z
x
y
Series or Parallel?
① Need only go through one beam to get from anchor → forcing pt. } ∴ parallel
or ② $y_{tot} = y_a = y_b$ ∴ parallel
$k_{tot} = k_a + k_b$
forcing pt.
$k_c \| k_c = \frac{k_c}{2}$
$k_c$
$\frac{k_c}{2} + \frac{k_c}{2} = k_c = k_{tot}$
anchor



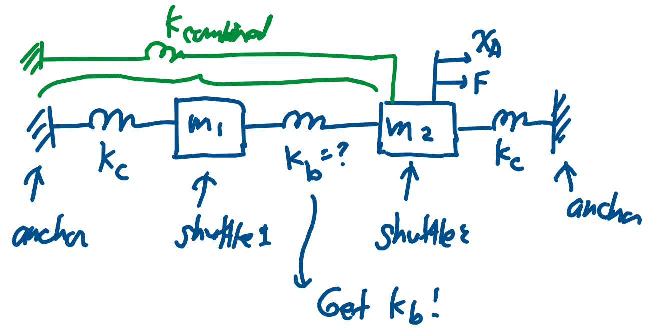

⇒ Find the stiffness at point A.

Apply F @ A → what is $x_A$?

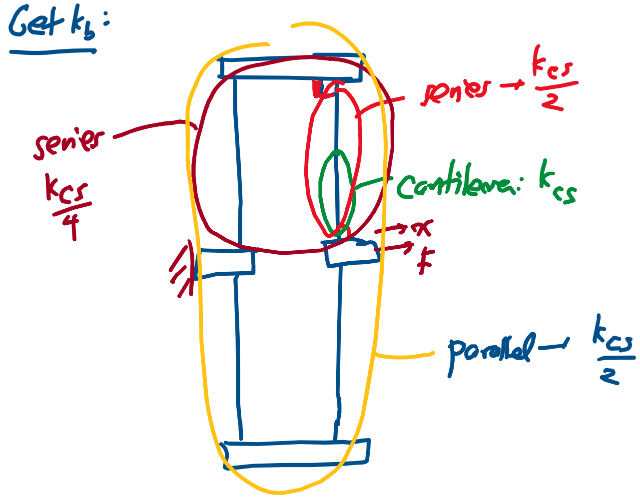
$k_A \triangleq$ stiffness @ point A ← want this

$x_A = \frac{F}{k_A}$

Assume shuttles & folding truss are rigid.



Get $k_b$:




$\therefore k_A = k_c + k_{combined}$

$= k_c + k_c \| k_b = k_c + k_c \| \frac{k_{cs}}{2} = k_A$

where $k_c = 2Eh\left(\frac{W}{L_c}\right)^3$

$k_{cs} = 2Eh\left(\frac{W}{L_{cs}}\right)^3$

## Tensioned Spring (Non-Ideality)

- Important case for MEMS suspensions, since the thin films comprising them are often under residual stress
- Consider small deflection case: y(x) « L

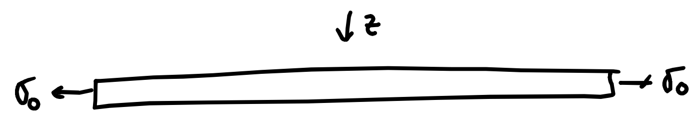

Governing differential equation: (Euler Beam Equation)

$$EI_z \frac{d^4y}{dx^4} - S\frac{d^2y}{dx^2} = F\delta(x-L)$$

Axial Load (S); Unit impulse @ x=L ($F\delta(x-L)$)

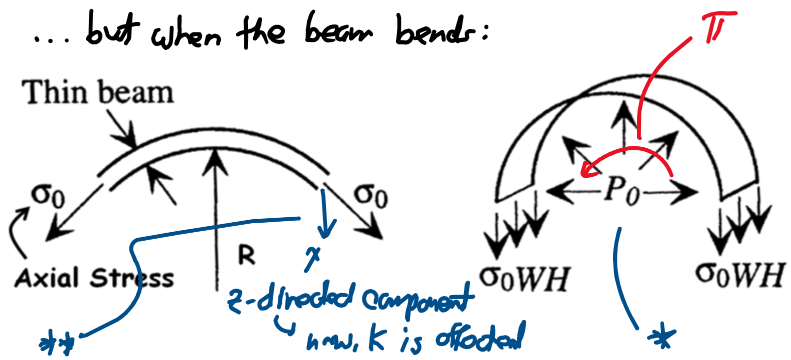
## Heuristic Derivation for the Euler Beam Equation

Consider first a straight beam under an axial stress:




⇒ no effect on z-directed stiffness when the beam is straight

... but when the beam bends:



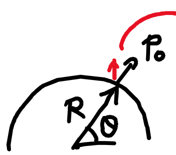
* ( Upward pressure $P_o$ to counteract the downward force from ** to keep everything in static equilibrium

For ease of analysis:

Assume the beam bends to an angle $\pi$

↳ Downward vertical force: $2\sigma_o WH$

Get upward force due to $P_o$:




$P_z(\theta) = P_o \sin\theta$

$$F_u = \int_0^{\pi} (P_o \sin\theta) W (R\, d\theta) = -P_o W R \cos\theta \Big|_0^{\pi} = 2RWP_o$$

[Equilibrium] → $2RWP_o = 2\sigma_o WH \longrightarrow P_o = \dfrac{\sigma_o H}{R}$

$\left[ q_o = \dfrac{\text{beam load}}{\text{unit length}} = P_o W, \quad \dfrac{1}{R} = \dfrac{d^2 w}{dx^2} \right]$ ← beam displacement

$$q_o = \sigma_o WH \frac{d^2 w}{dx^2}$$

← generalizes to the case of small displacements & angles

Using the Differential Beam Bending Eq.

$$\frac{d^2 w}{dx^2} = -\frac{M}{EI} \xrightarrow{???} \frac{d^4 w}{dx^4} = \frac{q}{EI}$$

← load/unit length

* Relationships Between Forces & Moments on a Fully-Loaded Differential Beam Element