Lecture 15: Beam Combos II

Announcements:
- HW#4 online, due Tuesday, 3/19, 9 a.m.
- Midterm Exam: Thursday, March 21, 11-12:30 p.m., 293 Cory (right here)
- No lecture next Tuesday, 3/19
  - The EECS Faculty Retreat is this day
  - I will post a video lecture instead

Reading: Senturia, Chpt. 9

Lecture Topics:
- Bending of beams
- Cantilever beam under small deflections
- Combining cantilevers in series and parallel
- Folded suspensions
- Design implications of residual stress and stress gradients

Last Time:
- Spring circuits
- Continue with this

Typical Questions: all demand knowledge of $x = f(F)$

1. How does the structure move in response to a force at a specific location? S in turn requires
2. What is the frequency response to an AC force that applied at a specific location?
3. Noise?
4. Response to environmental stimuli? (e.g., rotation)
5. How does stress affect the behavior of the structure?
Procedure:
1. Build the clt. (extract the clt.)

2. Analyze to get \( x_1 \), \( f(F) \) 

Anchor

\[
\begin{align*}
\chi_{tot} &= \chi_1 + \chi_2 = F \left( \frac{1}{k_1} + \frac{1}{k_2} \right) = \frac{F}{k_1/k_2} = \frac{F}{k_{tot}} \\
\chi_1 = \frac{F}{k_1}, \chi_2 = \frac{F}{k_2} \\
\text{"II" operator} &= \frac{AA}{AB} = \frac{AB}{A+B} \\
\end{align*}
\]

\[
\begin{align*}
k_{tot} &= k_1 \cdot k_2 \quad (\text{for } k_1 + k_2 \text{ in series}) \\
\text{For } EE's: \quad \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{C_{tot}} \quad \text{capacitors combine like parallel} \\
\text{parallel if can go from one to another}
\end{align*}
\]

\[
\begin{align*}
\chi_{tot} &= \chi_1 = \chi_2 \\
F_2 &= k_2 \chi_{tot} \\
\chi_1 &= \chi_2
\end{align*}
\]

Another way to tell that springs are in parallel:

\[
\begin{align*}
F &= F_1 + F_2 = (k_1 + k_2) \chi_{tot} \\
k_{tot} &= k_1 + k_2 \\
\end{align*}
\]
Series Combination of Springs

Anch (Clamped)

\[ L = 2L_c \]

Guided

\[ y_1 \]

\[ y(L) \]

\[ \sum y = y_1 + y_2 \]

Series

or must go thru both springs \( \rightarrow \) series

\[ y_{tot} = y_1 + y_2 \]

\[ k_{tot} = k_1 + k_2 \]

Guided B.C.

\[ \frac{1}{k_1} + \frac{1}{k_2} = \frac{k_1 k_2}{k_1 + k_2} \]

Clamped B.C.

\[ k_{tot} = k_1 k_2 \]

\[ k_{tot} = k_1 + k_2 \]

Series vs Parallel?

1. Need only go through one beam \( \rightarrow \) parallel

2. Go from anch to forcing pt. \( \rightarrow \) series

\[ y_{tot} = y_a, y_b \]

\[ k_{tot} = k_{tot} \]

\[ k_{tot} = k_{tot} \]

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\[ F = \frac{3EIz}{L^2} = \frac{1}{2} Eh \left( \frac{w^3}{3} \right) \]

\[ k_c = \frac{2EIz}{(Lz)^3} = \frac{2Eh}{L^3} \left( \frac{w^3}{3} \right) \]

\[ k_c = 4 \left( \frac{k_c}{2} \right) : k_c \]

\[ x_0 = \frac{L}{4} \]

\[ F = \frac{L}{2} \]

\[ \text{full beam length} \]

⇒ Find the stiffness at point A.

⇒ Apply forces at what is & nxA: \[ nxA = \frac{E}{k_A} \]

k_A: stiffness at point A

Assume shutter & folding truss are rigid.

\[ k_c \]

\[ k_b = \frac{F}{x_0} \]

\[ \text{anchored} \]

\[ \text{shutter 1} \]

\[ \text{shutter 2} \]

\[ \text{anchored} \]
Get $k_b$:

- $k_s$ is the spring constant for the thin film.
- $k_{cs}$ is the additional spring constant due to residual stress.

\[ k_A = k_c + k_{\text{combined}} = k_c + k_{cs} \]

\[ k_{cs} = \frac{k_{cs}}{2} \]

\[ k_A = k_c + \frac{k_{cs}}{2} \]

**Tensioned Spring (Non-ideality)**

- Important case for MEMS suspensions, since the thin films comprising them are often under residual stress.
- Consider small deflection case: $y(x) \ll L$

Governing differential equation: (Euler Beam Equation)

\[ EI_z \frac{d^4y}{dx^4} - S \frac{d^2y}{dx^2} = F \delta(x - L) \]

Axial Load \ Unit impulse @ $x = L$

**Heuristic Derivation for the Euler Beam Equation**

Consider first a straight beam under an axial stress:

\[ \sigma_0 \]

\[ \text{no effect on 2-direction stiffness when the beam is straight} \]

... but when the beam bends:

Thin beam

- Axial Stress $\sigma_0$
- 2-Directed Component $\sigma_0 WH$
- $k$ is affected
* Upward pressure \( P_0 \) to counteract the downward force from \( \sigma_0 W H \) to keep everything in static equilibrium.

For ease of analysis:
Assume the beam bends at an angle \( \Theta \)

\[
\text{Downward vertical force: } 2 \sigma_0 W H \]

Get upward force due to \( P_0 \):

\[
P_2(\Theta) = P_0 \sin \Theta
\]

\[
F_u = \int_0^\Theta (P_0 \sin \Theta) W (R \Theta) \, d\Theta
= -P_0 W R \cos \Theta \big|_0^\Theta
= 2 R W P_0
\]

\[\text{[Equilibrium]} \quad 2 R W P_0 = 2 \sigma_0 W H \rightarrow P_0 = \frac{\sigma_0 H}{R}\]

\[q_0 = \text{beam load/unit length} = \frac{P_0 W}{R} \quad \frac{d^3 w}{d x^3} \text{ beam displacement}
\]

\[q_0 = \frac{\sigma_0 W H}{R} \frac{d^3 w}{d x^3} \rightarrow \text{generalizes to the case of small displacement of angles}\]

Using the differential beam bending Eq. 102:

\[
\frac{d^3 w}{d x^3} = -\frac{M}{EI} \quad \text{and} \quad \frac{d^3 w}{d x^3} = \frac{q}{EI}
\]

* Relationship Between Forces & Moments on a Fully-Loaded Differential Beam Element.