Lecture 16: Energy Methods

- Announcements:
- HW#4 online, due Tuesday, 3/19, 9 a.m.
- Midterm Exam: Thursday, March 21, 11-12:30 p.m., 293 Cory (right here)
  - Passed out old exams
  - Went through Midterm Info Sheet
- No lecture next Tuesday, 3/19
  - The EECS Faculty Retreat is this day
  - I will post a video lecture instead
  - Kyle will hold a Review Session for the Exam during this lecture period
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- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients
- -----------------------------------
- Reading: Senturia, Chpt. 10
- Lecture Topics:
  - Energy Methods
  - Virtual Work
  - Energy Formulations
  - Tapered Beam Example
- -----------------------------------
- Last Time: Tensioned spring analysis
- Continue with this

Tensioned Spring (Non-Ideality)

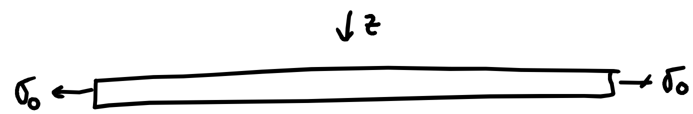
- Important case for MEMS suspensions, since the thin films comprising them are often under residual stress
- Consider small deflection case: y(x) « L

Governing differential equation: (Euler Beam Equation)

$$EI_z \frac{d^4 y}{dx^4} - \underset{\text{Axial Load}}{S} \frac{d^2 y}{dx^2} = \underbrace{F\delta(x-L)}_{\text{Unit impulse @ x=L}}$$

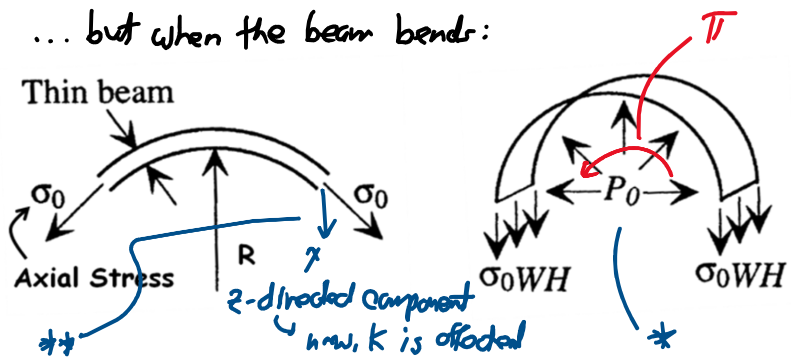
Heuristic Derivation for the Euler Beam Equation

Consider first a straight beam under an axial stress:



⇒ no effect on z-directed stiffness when the beam is straight

... but when the beam bends:



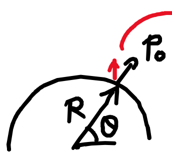
* ↳ Upward pressure P_o to counteract the downward force from ** to keep everything in static equilibrium

For ease of analysis:

Assume the beam bends to an angle π

↳ Downward vertical force: $2\sigma_o W H$

Get upward force due to P_o:



$P_z(\theta) = P_o \sin\theta$

$$F_u = \int_0^{\pi} (P_o \sin\theta) W (R\,d\theta)$$

$$= -P_o W R \cos\theta \Big|_0^{\pi}$$

$$= 2RWP_o$$

[Equilibrium] → $2RWP_o = 2\sigma_o W H \longrightarrow P_o = \dfrac{\sigma_o H}{R}$

$\left[q_o = \dfrac{\text{beam load}}{\text{unit length}} = P_o W, \quad \dfrac{1}{R} = \dfrac{d^2 w}{dx^2} \right]$ beam displacement

$$q_o = \sigma_o W H \frac{d^2 w}{dx^2}$$

← generalizes to the case of small displacements & angles

*d

Using the Differential Beam Bending Eq.

$$+\ -\frac{d^2 w}{dx^2} = -\frac{M}{EI} \xrightarrow{???} \frac{d^4 w}{dx^4} = \frac{q}{EI}$$

load/unit length

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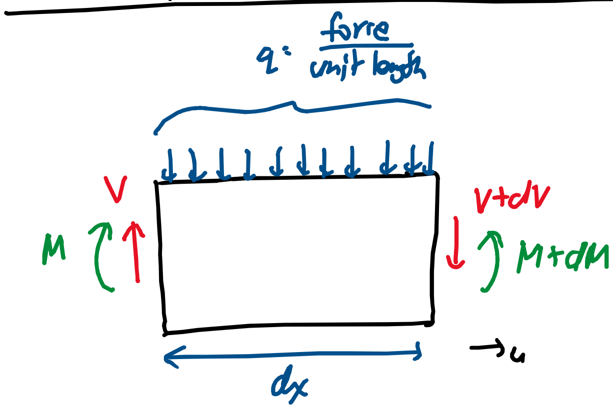
* Relationships Between Forces & Moments on a Fully-Loaded Differential Beam Element



[Total Static Equilibrium] ⇒ total force = 0

F_T = total force $= q\,dx + (\cancel{V} + dV) - \cancel{V} = 0$

$$\therefore \boxed{\frac{dV}{dx} = -q} \qquad (1)$$

⇒ also, total moment w/r to left-hand edge = 0

$$M_T = (\cancel{M} + dM) - \cancel{M} - (V + \cancel{dV})dx - \tfrac{1}{2} q\, \cancel{dx^2} = 0$$

[neglect products of differentials]

$$\int_0^{dx} (q\,du)\,u = \tfrac{1}{2} q\, dx^2$$

$$dM - V\,dx = 0 \rightarrow \boxed{\frac{dM}{dx} = V} \qquad (2)$$

Using (1) & (2):

$$\left[\frac{d^2M}{dx^2} = \frac{dV}{dx} = -q\right] \quad + \quad ++$$

external load

$$EI\frac{d^4w}{dx^4} = q + q_0 \quad +$$

equiv. load from axial stress

$$\left[q_0 = \sigma_0 WH \frac{d^2w}{dx^2}\right]$$

$$\boxed{EI\frac{d^4w}{dx^2} - \underbrace{(\sigma_0 WH)}_{\text{tension in beam} = S}\frac{d^2w}{dx^2} = q}$$

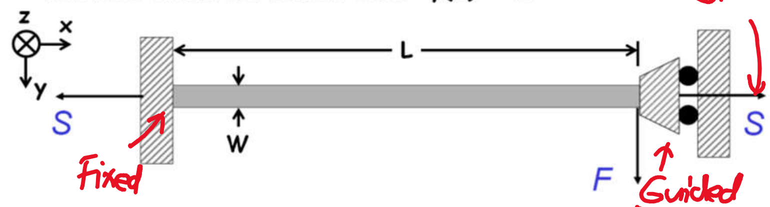
force

Euler Beam Equation

Clamped-Guided Beam Under Axial Load

- Important case for MEMS suspensions, since the thin films comprising them are often under residual stress
- Consider small deflection case: $y(x) \ll L$

Stress



Governing differential equation: (Euler Beam Equation)

$$EI_z\frac{d^4y}{dx^4} - S\frac{d^2y}{dx^2} = \underbrace{F\delta(x-L)}_{\text{Unit impulse @ x=L}}$$

Force

Axial Load

Need to solve this, then find the stiffness against this force (@ this location)

- Can solve the ODE using standard methods
  - Senturia, pp. 232-235: solves ODE for case of point load on a clamped-clamped beam (which defines B.C.'s)
  - For solution to the clamped-guided case: see S. Timoshenko, Strength of Materials II: Advanced Theory and Problems, McGraw-Hill, New York, 3rd Ed., 1955
- Result from Timoshenko: (Note: These include both bending & stretching)

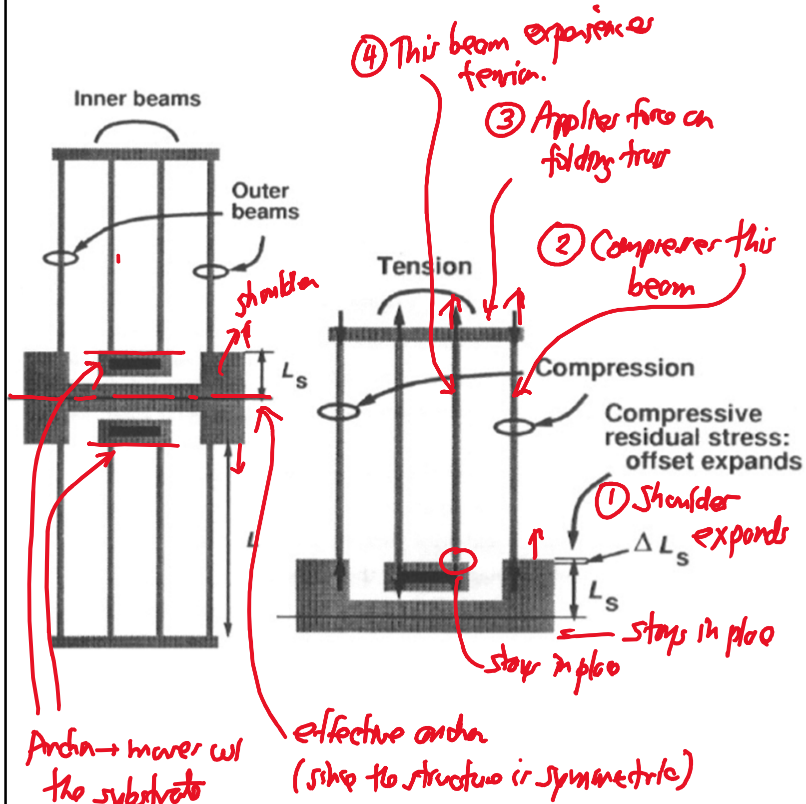
$S > 0$ (tension) $\qquad k^{-1} = \dfrac{pL - 2\tanh(pL/2)}{p|S|} = \dfrac{y(x=L)}{F}$

$S < 0$ (compression)

$$k^{-1} = \frac{-pL + 2\tan(pL/2)}{p|S|} = \frac{y(x=L)}{F}$$

where $p = \sqrt{\dfrac{|S|}{EI_z}}$ $\qquad$ To use, must know

Folded Beams Are Not Perfect



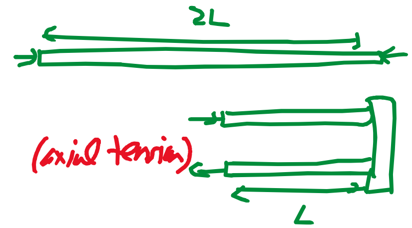
Get S:

① If the polySi structural material stress is ϵ_r, then the shoulder expands $\Delta L_s = \epsilon_r L_s$

② This then applies a load to the beams, $\Delta L = \Delta L_s$.

③ Beam Stress:

$$\epsilon_b = \frac{\Delta L}{2L} = \frac{\Delta L_s}{2L} = \pm \epsilon_r \frac{L_s}{2L}$$



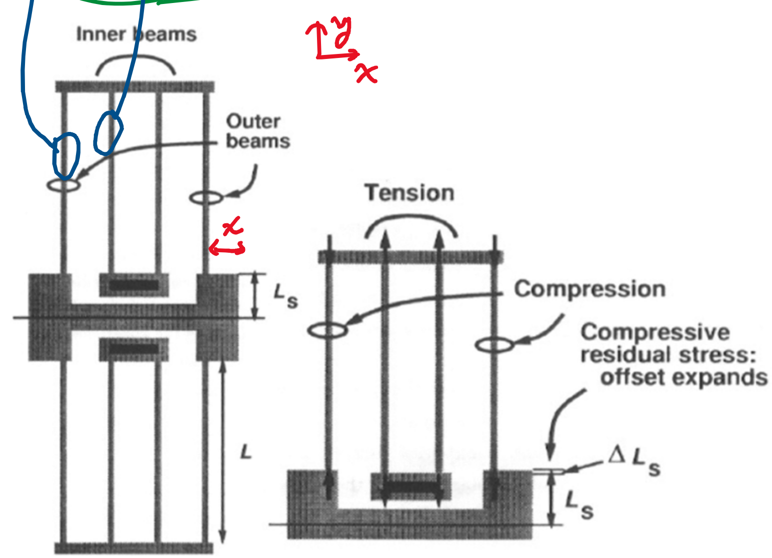
Stress Force:

$$S = \pm E \epsilon_r \left(\frac{L_s}{2L}\right) W h \quad \text{(axial tension)}$$

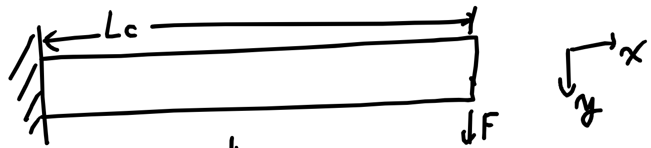
④ Spring Constants: 4 in parallel

$$k = 4\left(k_{com}^{-1} + k_{ten}^{-1}\right)^{-1} \quad : k_{com} \,||\, k_{ten} \text{ — series}$$

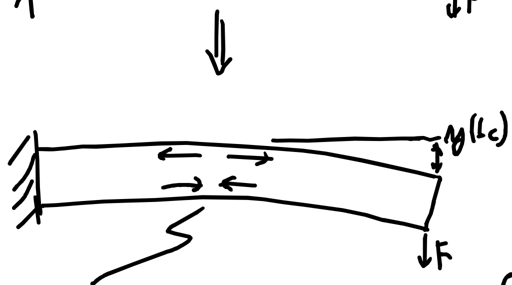
$$k = 4\left[\frac{-pL + 2\tan(pL/2)}{p|S|} + \frac{pL - 2\tanh(pL/2)}{p|S|}\right]^{-1}$$



Same Problem as Before: Take a beam, apply a force.



① Apply force.



② Beam responds by bending.

③ This force has done work:

$W = F \cdot y(L_c)$

④ Strain generated.

So the beam has received an influx of stored energy.

magnitude of " " determined by shape.

⑤ Then

U = Stored Energy − Work Done $\longrightarrow 0$

When we choose the right shape.

This is how we get the beam's response to F!

transfer function

$y(x) = f(x)$