Lecture 17: Resonance Frequency

Announcements:
- This is a video lecture, since I am at the EECS Faculty Retreat this day
- Module 9 on Energy Methods online
- Module 10 on Resonance Frequency online
- Midterm Exam: Thursday, March 21, 11-12:30 p.m., 293 Cory (right here)

Reading: Senturia, Chpt. 10
Lecture Topics:
- Energy Methods
- Virtual Work
- Energy Formulations
- Tapered Beam Example

Reading: Senturia, Chpt. 10: §10.5, Chpt. 19
Lecture Topics:
- Estimating Resonance Frequency
- Lumped Mass-Spring Approximation
- ADXL-50 Resonance Frequency
- Distributed Mass & Stiffness
- Folded-Beam Resonator
- Resonance Frequency Via Differential Equations

Last Time:
- Working through energy methods
- Continue with this ...

Some problem as before: Take a beam, apply a force.

1. Apply force.
2. Beam responds by bending.
3. This force has done work:
   \[ W = F \cdot y(l_c) \]
   where \( l_c \) is the beam's deflection.
4. Show that:
   \[ \frac{dU}{dt} = F \cdot y(l_c) \]
   so the beam has received an influx of stored energy. The magnitude of \( \frac{dU}{dt} \) is determined by shape.
5. Then
   \[ U = \text{Stored Energy} - \text{Work Done} \rightarrow 0 \]

[Diagram of a beam with forces and deflections]

[Red text]: Transfer function
\[ y(t) : h(s) \]
This is how we get the beam's response to \( F \).
Fundamentals: Energy Density

General Definition of Work:

\[ W(q) = \int_0^q e(q) \, dq \quad e = \text{effort} \]

For EE:

\[ W(q) = \int_0^q \frac{\partial q}{C} \, dq \]

Strain Energy Density

\[
\sigma(x) = E\varepsilon_x \\
\varepsilon_x = \frac{\partial u_x}{\partial x} \\
\sigma(x) \rightarrow \text{relates strain to strain} \\
\text{at position (x, y, z)}
\]

\[
\varepsilon_x = -\frac{1}{E} \frac{\partial u_x}{\partial x}
\]

Total Strain Energy:

\[
W = \iiint \left( \frac{1}{2} \varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2 \right) + \frac{1}{2} C (\partial u_y^2 + \partial u_z^2 + \partial u_\psi^2) \, dV
\]

Bending Energy Density

\[
W = \text{width into the plane} \\
\text{Neutral Axis} \quad \text{same as z before} \\
y(x) = \text{transverse displacement of neutral axis}
\]

First, find the bending energy \( W \) in an infinitesimal length \( dx \)

\[
dW_{\text{bend}} = W dx \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{2} E \varepsilon_x^2 (y) \, dy' \\
\varepsilon_x = \frac{y'}{R} \\
\varepsilon_x (y) = \frac{y' d^2 y}{d^2 x}
\]

\[
dW_{\text{bend}} = W dx \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{2} E \left[ \frac{d^2 y}{d^2 x} \right]^2 \, dy' \\
= \frac{1}{2} E \left( \frac{w h^3}{12} \right) \left( \frac{d^2 y}{d^2 x} \right)^2 \, dx \\
\frac{I_x}{E}
\]

\[
W_{\text{bend}} = \frac{1}{2} EI_x \int_0^L \left( \frac{d^2 y}{dx^2} \right)^2 \, dx
\]
Energy Due to Axial Load

\[ ds = \sqrt{(dx)^2 + (dy)^2} = dx \left[ 1 + \left(\frac{dy}{dx}\right)^2 \right]^{\frac{1}{2}} \]

Binomial theorem: \( \approx dx \left[ 1 + \frac{1}{2} \left(\frac{dy}{dx}\right)^2 \right] \)

\[ x = \frac{ds \cdot dx}{dx} = \frac{1}{2} \left(\frac{dy}{dx}\right)^2 \]

\[ dW_{\text{Axial}} = S \cdot x \cdot dx = \frac{1}{2} S \left(\frac{dy}{dx}\right)^2 dx \]

Axial Strain Energy

\[ W_{\text{Axial}} = \frac{1}{2} S \int_0^L \left(\frac{dy}{dx}\right)^2 dx \]

Estimating Resonance Frequency

\[ \frac{d^2x}{dt^2} = x(t) = x_0 \cos \omega t \]

Potential Energy

\[ W(t) = \frac{1}{2} k x^2(t) = \frac{1}{2} k x_0^2 \cos^2 \omega t \]

Kinetic Energy

\[ K(t) = \frac{1}{2} m \dot{x}(t)^2 = \frac{1}{2} m x_0^2 \omega^2 \sin^2 \omega t \]

\[ \dot{x} = \frac{dx}{dt} \] velocity

Remarks:
1. Energy must be conserved.
2. Total Energy = Potential Energy + Kinetic Energy at all times and locations on the structure

\[ W_{\text{max}} = \frac{1}{2} K x_0^2 = K_{\text{max}} = \frac{1}{2} m \omega^2 x_0^2 \]

\[ \omega \] peak displacement

maximum potential energy

maximum kinetic energy
[Image of a page from a lecture on MEMS Design, focusing on the resonance frequency of a device. The page includes mathematical equations and diagrams illustrating the design of a MEMS device. The text discusses the proof mass of the ADXL-50 and its suspension beams, noting the mass of the suspension beams can be ignored to simplify the analysis. It also mentions the suspension beam parameters and the fabrication process, highlighting the need for tension in the beams to avoid compression and the large deflection associated with it.]
Bending Contribution

\[ k_b = k_{s1} k_{c1} \left( \frac{1}{k_c} + \frac{1}{k_c} \right) = \frac{k_s}{2} = \frac{1}{2} \frac{3Ebh^3/12}{(L/2)^3} \]

\[ \Rightarrow k_s = EW \left( \frac{L}{2} \right)^3 = 0.24 \text{ Nm} \]

Stretching Contribution

\[ F_b = S \Delta l \frac{1}{2} \left( \frac{1}{k_c} + \frac{1}{k_c} \right) = \frac{1}{2} k_c \Delta l \]

Asymptotic small displacements

\[ k_{st} \approx \frac{E}{L} \]

\[ k_{st} = \frac{5}{L} \text{ Nm}^{-1} \]

Get the total spring constant

\[ k_{tot} = k_b + k_{st} = 4(0.24 + 0.88) = 4.5 \text{ Nm} \]

Now, get the resonant freq:

\[ f_0 = \frac{1}{2\pi} \sqrt{\frac{k_{tot}}{m}} = \frac{1}{2\pi} \sqrt{\frac{4.5}{162\times10^{-12} \text{ kg}}} \approx 2.65 \text{ kHz} \]

Apery's Data Sheet: \( f_0 = 24 \text{ kHz} \)

Capacitive transducer \( \rightarrow \) electrical stiffness

Find the Resonance Frequency When Mass and Stiffness Are Distributed

- Vibrating structure displacement function:

\[ y(x,t) = \hat{y}(x) \cos(\omega t) \]

Max displacement function (i.e., mode shape function), \( \hat{y}(x) \)

Seen when velocity \( y(x,t) = 0 \)

- Procedure for determining resonance frequency:
  - Use the static displacement of the structure as a trial function and find the strain energy \( W_{\text{max}} \) at the point of maximum displacement (e.g., when \( t = 0, \pi/\omega, \ldots \))
  - Determine the maximum kinetic energy when the beam is at zero displacement (e.g., when it experiences its maximum velocity)
  - Equate energies and solve for frequency

Get Maximum Kinetic Energy

Velocity:

\[ \mathbf{v}(x,t) = \frac{\partial y(x,t)}{\partial t} = -\omega \tilde{y}(x) \sin(\omega t) \]

Largest velocity

\[ y(x,t) = 0 \]

Velocity topographical mapping

When \( y(x,t) = 0 \), all the energy in the structure is kinetic \((W = 0)\)

\[ v(x, t = \frac{\pi}{2\omega}, \frac{3\pi}{2\omega}, \ldots) \]

\[ v(x, t = 2n\pi) = -\omega \tilde{y}(x) \]
Derive an expression for the resonance frequency of the above structure.

Maximum Kinetic Energy, $K_{\text{max}}$:

$$K_{\text{max}} = \int_0^L \frac{1}{2} \rho Wh dx \int [\tilde{y}(x)]^2 dx$$

To get frequency: $f_{\text{max}} = \frac{\omega}{2\pi}$

$$\omega = \sqrt{\frac{K_{\text{max}}}{\int_0^L \frac{1}{2} \rho Wh [\tilde{y}(x)]^2 dx}} \quad [\text{radians/s}]$$

$\omega$ = radian resonance freq.

$K_{\text{max}}$ = maximum potential energy

$\rho$ = density of the structural material

$W$ = beam width

$h$ = thickness

$\tilde{y}(x)$ = resonance mode shape

- Derive an expression for the resonance frequency of the above structure

Approximation:

$$\Rightarrow m = \text{shuttle mass} \quad \Rightarrow k = k_c$$

But not accurate enough in some applications.

For better accuracy, must integrate
Use the Rayleigh–Ritz Method: (energy method)

\[ K_{\text{max}} \leq \frac{1}{2} k x^2 \]

Find the kinetic energy term piece at a time

\[ K_{\text{max}} = K_f + K_t + K_b \]

shuttle, transverse beams

\[ = \frac{1}{2} M_s^2 \dot{x}_s + \frac{1}{2} M_t^2 \dot{x}_t + \frac{1}{2} \int M_b \ddot{x}_b \, dt \]

Velocity of the Shuttle:

\[ v_s = \omega x_0 \]

\[ \ddot{x}_s = \frac{1}{2} M_s \dddot{x}_0 \]

Velocity of Transverse:

\[ v_t = \frac{1}{2} \omega x_0 \]

\[ \ddot{x}_t = \frac{1}{2} \omega^2 x_0^2 \]

\[ \text{mass of both transverse} \]

\[ \text{velocities} \]

\[ \text{mass} \]

\[ \text{of both transverse} \]

\[ \text{segments} \]

\[ \dot{x}(y) = \frac{F_x}{4yEIT_2} \left( 3(y^2 - 2y^3) \right) , \; 0 \leq y \leq L \]

At \( y = L \): \[ \dot{x}(L) = \frac{x_0}{2} = \frac{F_x L^3}{8yEIT_2} \]

Substitute into (1):

\[ \dot{x}(y) = \frac{x_0}{2} \left[ 3 \left( \frac{y}{L} \right)^2 - 2 \left( \frac{y}{L} \right)^3 \right] \]

which yields for velocity:

\[ v_0(y) \Big|_{y=0} = \frac{x_0}{2} \left[ 3 \left( \frac{0}{L} \right)^2 - 2 \left( \frac{0}{L} \right)^3 \right] \omega_0 \]
Plugging into expression for \( K_b \):

\[
K_{[AB]} = \frac{1}{2} \int_0^L \frac{x_0^2 w_0^2}{4} \left[ 3 \left( \frac{y}{L} \right)^2 - 2 \left( \frac{y}{L} \right)^3 \right]^2 \, dx
\]

\[
= \frac{x_0^2 w_0^2 M_{[AB]}}{8L} \int_0^L \left[ 3 \left( \frac{y}{L} \right)^2 - 2 \left( \frac{y}{L} \right)^3 \right]^2 \, dy
\]

\( K_{[AB]} = \text{static mass per unit length} \)

For segment [CD]:

\[
K_{[CD]} = \frac{x_0^2 w_0^2 M_{[CD]}}{2L} \int_0^L \left[ 1 - \frac{3}{2} \left( \frac{y}{L} \right)^2 + \left( \frac{y}{L} \right)^3 \right] \, dy
\]

\[
K_{[CD]} = \frac{x_0^2 w_0^2 M_{[CD]}}{2L} \int_0^L \left[ 1 - \frac{3}{2} \left( \frac{y}{L} \right)^2 + \left( \frac{y}{L} \right)^3 \right] \, dy
\]

\( K_{[CD]} = \frac{82}{280} x_0^2 w_0^2 M_{[CD]} \)

Let \( M_b \) = total mass of all beams

Thus:

\[
K_b = 4L K_{[AB]} + 4L K_{[CD]} = \frac{6}{35} x_0^2 w_0^2 M_b
\]

\[
K_{[AB]} = \frac{x_0^2 w_0^2 M_{[AB]}}{8L} \int_0^L \left[ 3 \left( \frac{y}{L} \right)^2 - 2 \left( \frac{y}{L} \right)^3 \right]^2 \, dy
\]

\( K_{[AB]} = \text{static mass per unit length} \)

And:

\[
K_{max} = x_0^2 w_0^2 \left[ \frac{1}{2} M_s + \frac{1}{6} M_t + \frac{6}{35} M_b \right]
\]

\( K_{max} \) for the total mechanical load.

\[
W_{max} = \frac{1}{2} k x_0^2
\]

Then, using Rayleigh-Ritz:

\[
K_{max} = W_{max}
\]

\[
x_0^2 w_0^2 \left[ \frac{1}{2} M_s + \frac{1}{6} M_t + \frac{6}{35} M_b \right] = \frac{1}{2} k x_0^2
\]

\[
\omega_0 = \sqrt{\frac{k}{M_{eq}}}
\]

where \( M_{eq} = M_s + \frac{1}{6} M_t + \frac{12}{35} M_b \)

(Resonance Frequency of a Folded-Beam Suspended Shuttle)