Lecture 17: Resonance Frequency

- Announcements:
- This is a video lecture, since I am at the EECS Faculty Retreat this day
- Module 9 on Energy Methods online
- Module 10 on Resonance Frequency online
- Midterm Exam: Thursday, March 21, 11-12:30 p.m., 293 Cory (right here)
- ------------------------------------
- Reading: Senturia, Chpt. 10
- Lecture Topics:
  - Energy Methods
  - Virtual Work
  - Energy Formulations
  - Tapered Beam Example
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- Reading: Senturia, Chpt. 10: §10.5, Chpt. 19
- Lecture Topics:
  - Estimating Resonance Frequency
  - Lumped Mass-Spring Approximation
  - ADXL-50 Resonance Frequency
  - Distributed Mass & Stiffness
  - Folded-Beam Resonator
  - Resonance Frequency Via Differential Equations
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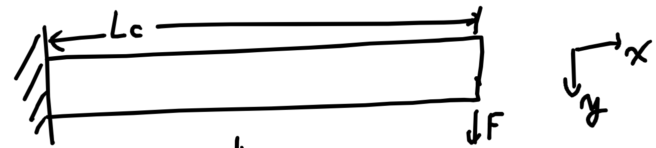
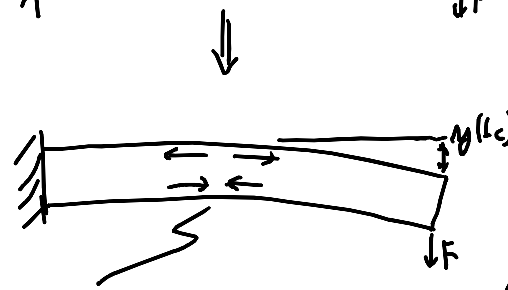
- Last Time:
- Working through energy methods
- Continue with this …

Same Problem as Before: Take a beam, apply a force.



① Apply force.



② Beam responds by bending.

③ This force has done work: $W = F \cdot y(L_c)$

④ Strain generated.

So the beam has received an influx of stored energy.

magnitude of " " determined by shape.

⑤ Then

U: Stored Energy - Work Done $\longrightarrow 0$

When we choose the right shape.

transfer function $y(x) = f(x)$

This is how we get the beam's response to F!

Fundamentals: Energy Density

General Definition of Work:

$$\mathcal{W}(q_1) = \int_0^{q_1} e(q)\,dq$$

q: displacement
e: effort

for EE: $\mathcal{W}(Q) = \int_0^Q \frac{Q}{C}\,dQ$

Strain Energy Density

Value of strain @ position (x, y, z)

$$w = \int_0^{\epsilon_x} \sigma_x\, d\epsilon_x$$

$\sigma_x(\epsilon_x) \rightarrow$ relates stress to strain @ position (x, y, z)

$[\sigma_x = E\epsilon_x]$

$$w = \int_0^{\epsilon_x} E\epsilon_x\, d\epsilon_x = \frac{1}{2}E\epsilon_x^2$$

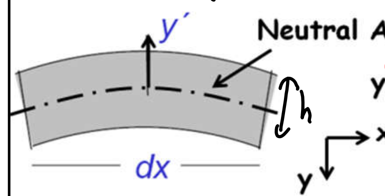
Total Strain Energy: [J]

volume

$$\mathcal{W} = \iiint\left(\frac{1}{2}E\left(\epsilon_x^2+\epsilon_y^2+\epsilon_z^2\right)+\frac{1}{2}G\left(\gamma_{xy}^2+\gamma_{xz}^2+\gamma_{yz}^2\right)\right)dV$$

Bending Energy Density

W = width into the plane



same as z before

y(x) = transverse displacement of neutral axis

First, find the bending energy $d\mathcal{W}_{bend}$ in an infinitesimal length dx

$$d\mathcal{W}_{bend} = W\,dx\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{2}E\epsilon_x^2(y')\,dy'$$

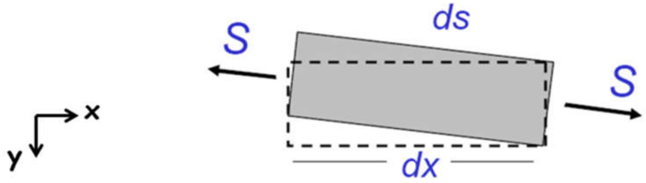
$$\left[\frac{1}{R} = \frac{d^2y}{dx^2},\ \epsilon_x = \frac{y'}{R}\right] \Rightarrow \epsilon_x(y') = y'\frac{d^2y}{dx^2}$$

$$d\mathcal{W}_{bend} = W\,dx\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{2}E\left[y'\frac{d^2y}{dx^2}\right]^2 dy'$$

$$= \frac{1}{2}E\underbrace{\left(\frac{Wh^3}{12}\right)}_{I_z}\left(\frac{d^2y}{dx^2}\right)^2 dx$$

$$\therefore \boxed{\mathcal{W}_{bend} = \frac{1}{2}EI_z\int_0^L\left(\frac{d^2y}{dx^2}\right)^2 dx}$$

Energy Due to Axial Load



⇒ energy related lengthening:

$$ds = \left[(dx)^2 + (dy)^2\right]^{1/2} = dx\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2}$$

binomial theorem

$$\simeq dx\left[1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2\right]$$

$$\therefore\ \epsilon_x = \frac{ds - dx}{dx} = \frac{1}{2}\left(\frac{dy}{dx}\right)^2$$

$$\left[dW_{axial} = S\epsilon_x dx = \frac{1}{2}S\left(\frac{dy}{dx}\right)^2 dx\right]$$

$$\boxed{W_{axial} = \frac{1}{2}S\int_0^L \left(\frac{dy}{dx}\right)^2 dx}$$

Axial Strain Energy

⇒ Look @ sheer strain energy in your module.

- Go through Module 9, slides 10-18

Estimating Resonance Frequency

$$\mapsto x(t) = X_o \cos\omega_o t$$

Potential Energy

$$W(t) = \frac{1}{2}kx^2(t) = \frac{1}{2}kX_o^2\cos^2\omega_o t$$

Kinetic Energy

$$K(t) = \frac{1}{2}m\dot{x}^2(t) = \frac{1}{2}mX_o^2\omega_o^2\sin^2\omega_o t$$

$\dot{x} = \frac{dx}{dt}$ = velocity

Remarks:

① Energy must be conserved.

② Total Energy = Potential Energy + Kinetic Energy at all times & locations on the structure

$$W_{max} = \frac{1}{2}kX_o^2 = K_{max} = \frac{1}{2}m\omega_o^2X_o^2$$

maximum potential energy

peak displacement

maximum kinetic energy

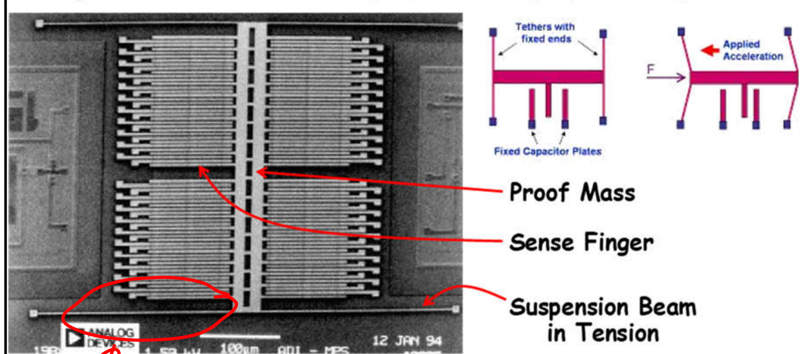
radian frequency

*

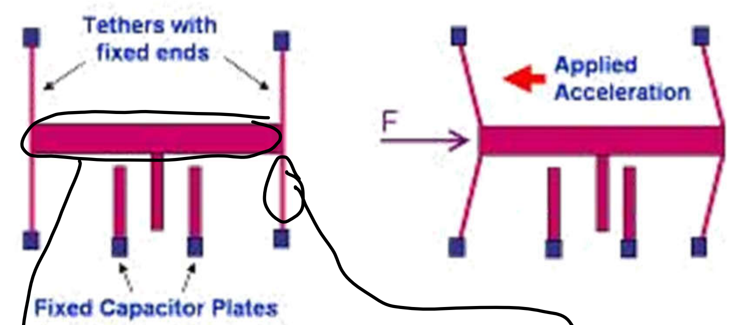
$$\boxed{\omega_0 = \sqrt{\frac{k}{m}}}$$

⇒ good for problems whose mass & stiffness can be separated
i.e., they are distinct

- The proof mass of the ADXL-50 is many times larger than the effective mass of its suspension beams
  - Can ignore the mass of the suspension beams (which greatly simplifies the analysis)
- Suspension Beam: L = 260 μm, h = 2.3 μm, W = 2 μm



In fabrication: purposely introduce a tensile stress in the beams!
a large one → why?
to avoid compression at all cost
buckling → dead device

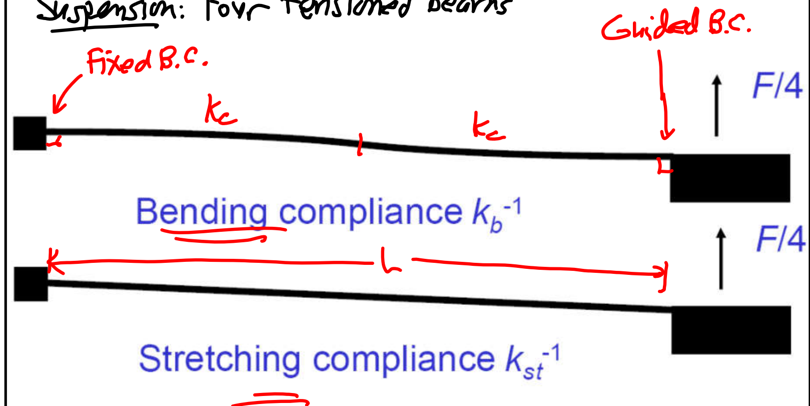


mass of structure >> mass of the springs
∴ ignore the mass of the springs

stiffness of the springs << stiffness of structure
∴ ignore the stiffness of the structure

For the ADXL-50, 60% the mass comes from the sense fingers → M = 162 ng

Suspension: Four tensioned beams

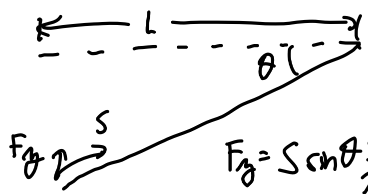


Bending Contribution

$$k_b = k_c \| k_c = \left(\frac{1}{k_c}+\frac{1}{k_c}\right)^{-1} = \frac{k_c}{2} = \frac{1}{2}\,\frac{3E(wh^3/12)}{(L/2)^3}$$

$$\Rightarrow k_b = Ew\left(\frac{h}{L}\right)^3 = 0.24\ \text{N/m}$$

Stretching Contribution



$$F_y = S\sin\theta \simeq S\left(\frac{y}{L}\right) = \underbrace{\left(\frac{S}{L}\right)}_{k_{st}} y$$

(assume small displacements)

k_{st} ↑ stretching stiffness

$$k_{st} = \frac{S}{L} = \frac{\sigma_r wh}{L} = 0.88\ \text{N/m}$$

Get the total spring constant

bending stiffness } parallel → add!
stretching stiffness

$$k_{tot} = 4(k_b+k_{st}) = 4(0.24+0.88) = \underline{4.5\ \text{N/m}}$$

Now, get the resonance freq:

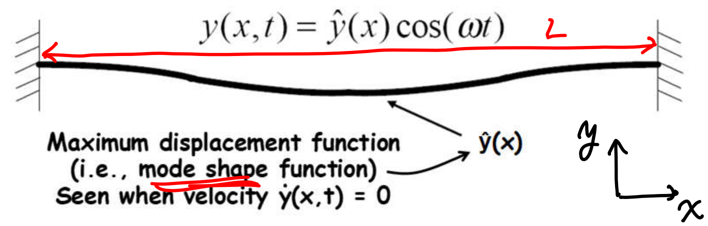
$$f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{4.48\ \text{N/m}}{162\times10^{-12}\ \text{kg}}} = 26.5\ \text{kHz}$$

ADXL-50 DataSheet: f_o = 24kHz ↔ difference?

↳ capacitive transducer → electrical stiffness

Find the Resonance Frequency When Mass & Stiffness Are Distributed

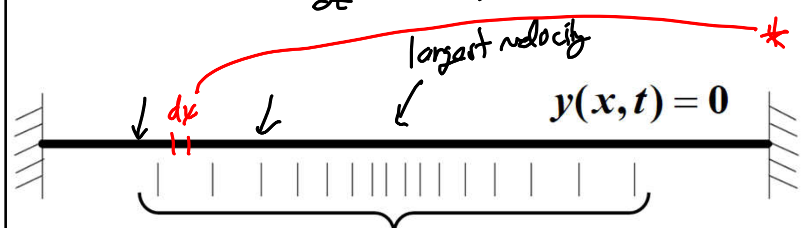
- Vibrating structure displacement function:



- Procedure for determining resonance frequency:
  - Use the static displacement of the structure as a trial function and find the strain energy W_{max} at the point of maximum displacement (e.g., when t=0, π/ω, …)
  - Determine the maximum kinetic energy when the beam is at zero displacement (e.g., when it experiences its maximum velocity)
  - Equate energies and solve for frequency

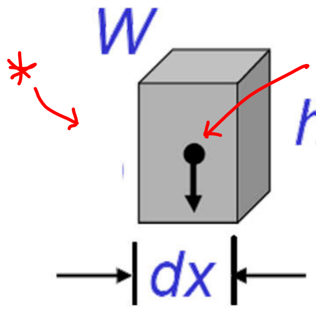
Get Maximum Kinetic Energy

Velocity: $v(x,t) = \frac{\partial y(x,t)}{\partial t} = -\omega\hat{y}(x)\sin\omega t$



when $y(x,t)=0$, all the energy in the structure is kinetic ($W=0$) ← $t = \frac{\pi}{2\omega}, \frac{3\pi}{2\omega}, \ldots$

↳ $v\left(x, \frac{(2m+1)\pi}{2\omega}\right) = -\omega\hat{y}(x)$



velocity: $v = -\omega \hat{y}(x)$ $\frac{(2m+1)\pi}{2W}$

$dK = \frac{1}{2} \cdot dm \cdot [v(x,t)]^2$

$dm = \rho(Wh\,dx)$ ← density

Maximum Kinetic Energy, K_{max}:

$$K_{max} = \int_0^L \frac{1}{2}\rho Wh\,dx\,v^2(x,t) = \int_0^L \frac{1}{2}\rho Wh\omega^2[\hat{y}(x)]^2 dx$$

To get frequency: $K_{max} = W_{max}$

$$\therefore \quad \omega = \sqrt{\frac{W_{max}}{\int_0^L \frac{1}{2}\rho Wh[\hat{y}(x)]^2 dx}} \quad [\text{radians/s}]$$

ω = radian resonance freq.

W_{max} = maximum potential energy

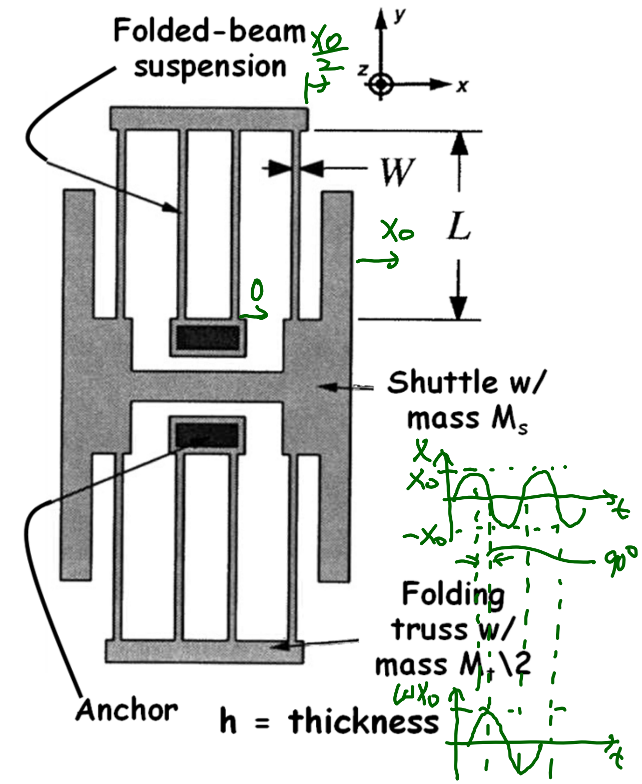
ρ = density of the structural material

W = beam width

h = " thickness

$\hat{y}(x)$ = resonance mode shape

Resonance Freq. of a Folded Beam Structure



- Derive an expression for the resonance frequency of the above structure

Approximation)

⇒ m = shuttle mass
⇒ $k = k_c$

$\omega_o = \sqrt{\frac{k_c}{m}}$ ← But not accurate enough for some applications.

⇒ for better accuracy, must integrate

Use the Rayleigh-Ritz Method: (energy method)

$$\mathcal{K}_{max} = \mathcal{W}_{max} \leftarrow \frac{1}{2}kx^2$$

Find the kinetic energy → one piece at a time

$$\mathcal{K}_{max} = \underbrace{\mathcal{K}_s}_{\text{shuttle}} + \underbrace{\mathcal{K}_t}_{\text{truss}} + \underbrace{\mathcal{K}_b}_{\text{beams}}$$

$$= \frac{1}{2}v_s^2 M_s + \frac{1}{2}v_t^2 M_t + \frac{1}{2}\int v_b^2\, dM_b$$

Velocity of the Shuttle: $v_s = \omega_0 X_0$

↑ res. freq. ↑ maximum displacement of shuttle

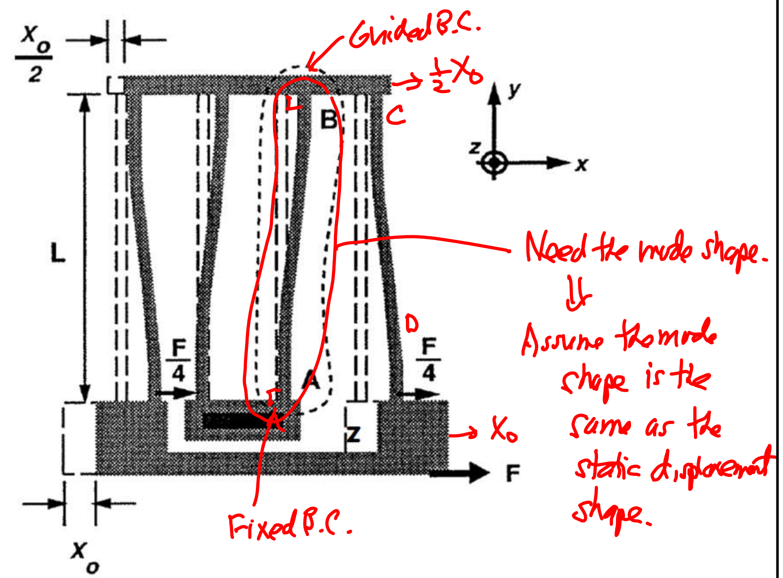
$$\therefore \mathcal{K}_s = \frac{1}{2}v_s^2 M_s = \boxed{\frac{1}{2}\omega_0^2 X_0^2 M_s = \mathcal{K}_s}$$

Velocity of Truss: $v_t = \frac{1}{2}v_s = \frac{1}{2}\omega_0 X_0$

$$\therefore \mathcal{K}_t = \frac{1}{2}\left(\frac{1}{2}\omega_0 X_0\right)^2 M_t = \boxed{\frac{1}{8}\omega_0^2 X_0^2 M_t = \mathcal{K}_t}$$

mass of both trusses

Velocity of the Beam Segments: → first beam [AB]



Segment [AB]:

$$\hat{x}(y) = \frac{F_x}{48EI_z}\left(3Ly^2 - 2y^3\right), \quad 0 \le y \le L \qquad (1)$$

$$\text{At } y = L: \; x(L) = \frac{X_0}{2} = \frac{F_x L^3}{48EI_z} \leftarrow \text{B.C.}$$

Substitute into (1):

$$\hat{x}(y) = \frac{X_0}{2}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]$$

which yields for velocity:

$$v_b(y)\Big|_{[AB]} = \frac{X_0}{2}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]\omega_0$$

Plugging into expression for $\mathcal{K}_b$:

$$\mathcal{K}_{[AB]} = \frac{1}{2}\int_0^L \frac{X_o^2\omega_o^2}{4}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]^2 dM_{[AB]}$$

$$= \frac{X_o^2\omega_o^2}{8}\frac{M_{[AB]}}{L}\int_0^L\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]^2 dy$$

mass per unit length

$M_{[AB]}$ = static mass

$$\boxed{\mathcal{K}_{[AB]} = \frac{13}{280}X_o^2\omega_o^2 M_{[AB]}}$$

For segment [CD]:

$$v_b(y)\Big|_{[CD]} = X_o\left[1 - \frac{3}{2}\left(\frac{y}{L}\right)^2 + \left(\frac{y}{L}\right)^3\right]\omega_o$$

Thus:

$$\mathcal{K}_{[CD]} = \frac{X_o^2\omega_o^2 M_{[CD]}}{2L}\int_0^L\left[1 - \frac{3}{2}\left(\frac{y}{L}\right)^2 + \left(\frac{y}{L}\right)^3\right]^2 dy$$

$$\boxed{\mathcal{K}_{[CD]} = \frac{83}{280}X_o^2\omega_o^2 M_{[CD]}}$$

static mass of beam [CD]

Let $M_b \triangleq$ total mass of all 8 beams

Thus:

$$\mathcal{K}_b = 4\mathcal{K}_{[AB]} + 4\mathcal{K}_{[CD]} = \frac{6}{35}X_o^2\omega_o^2 M_b$$

and

$$\boxed{\mathcal{K}_{max} = X_o^2\omega_o^2\left[\frac{1}{2}M_s + \frac{1}{8}M_t + \frac{6}{35}M_b\right]}$$

both trusses; all beams; for the total mechanical ckt.

W_{max} → max. potential energy → equal to the work done to achieve maximum deflection

$$W_{max} = \frac{1}{2}k_x X_o^2$$

Then, using Rayleigh-Ritz:

$$\mathcal{K}_{max} = W_{max}$$

$$\cancel{X_o^2}\omega_o^2\left[\frac{1}{2}M_s + \frac{1}{8}M_t + \frac{6}{35}M_b\right] = \frac{1}{2}\cancel{k_x}^{\,k_c}\cancel{X_o^2}$$

$$\boxed{\omega_o = \left[\frac{k_c}{M_{eq}}\right]^{1/2} \quad \text{where } M_{eq} = M_s + \frac{1}{4}M_t + \frac{12}{35}M_b}$$

(Resonance Freq. of a Folded-Beam Suspended Shuttle)