Lecture 19: Capacitive Transducers

- **Announcements**:
- Module 12 on Capacitive Transducers online
- HW#5 online for a while; due Tuesday, 4/9
- Project Definition online (discussed last time)
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- Reading: Senturia, Chpt. 5, Chpt. 6
- Lecture Topics:
  - Energy Conserving Transducers
    - Charge Control
    - Voltage Control
  - Parallel-Plate Capacitive Transducers
    - Linearizing Capacitive Actuators
    - Electrical Stiffness
  - Electrostatic Comb-Drive
    - 1^{st} Order Analysis
    - 2^{nd} Order Analysis
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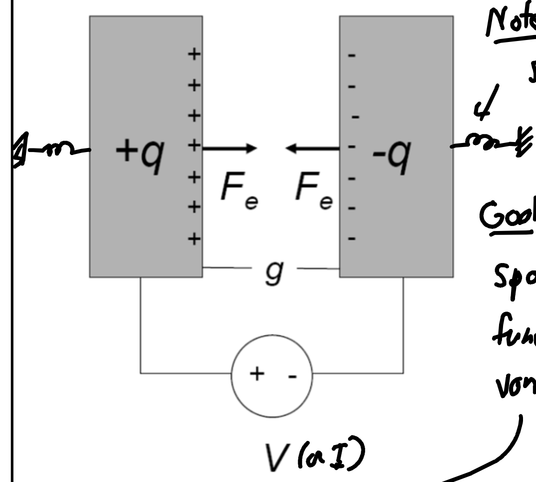
- **Last Time**:
- Finished our first pass on equivalent circuits
- Now, start on capacitive transducers …

Basic Physics of Electrostatic Actuation



Note: Assume plates supported elastically.

Goal: Determine gap spacing g as a function of input variables.

1^{st}: Determine the energy of the system.

2^{nd}: Ask: What can I do to change the energy of the system?

1. change the charge q
2. change the separation g

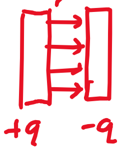
$$\Delta W(q,g) = V\Delta q + F_e \Delta g$$

$$dW = Vdq + F_e dg$$

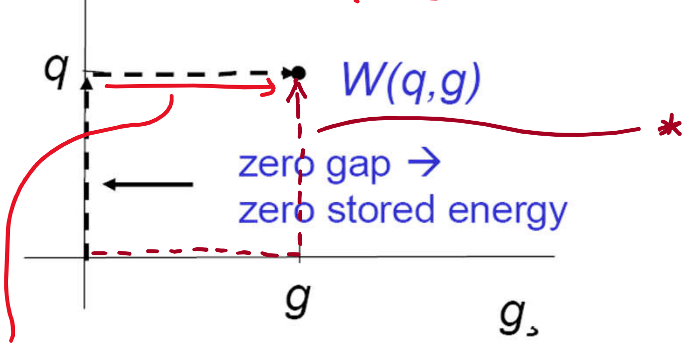
hold q = const. $\rightarrow Vdq \rightarrow 0$

$$dW = F_e dg \rightarrow F_e = \left.\frac{dW}{dg}\right|_{q=const.}$$

Stored Energy



$\mathcal{E} = \frac{q}{\epsilon A}$ ← cross-sectional area



No change in charge: $dq = 0$

$W = 0 + \int_0^g F_e dg'$

$F_e = \left(\frac{q}{2}\right)\mathcal{E} = \frac{1}{2}\frac{q^2}{\epsilon A}$ (independent of g)

$\therefore W = \int_0^g F_e dg' = F_e g' \Big|_0^g = F_e g$

$$\boxed{W(g) = \frac{1}{2}\frac{q^2}{\epsilon A} g}$$ — **

* — Work done to charge C to q at a fixed gap g.

$dW = V dq + F_e \cancel{dg}^{\,0}$

For a capacitor:

$q = CV \rightarrow V = \frac{q}{C}$

$\therefore W(q) = \int_0^q V dq = \int_0^q \left(\frac{q}{C}\right) dq = \frac{1}{2}\frac{q^2}{C}$

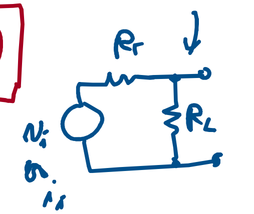
$$= \boxed{\frac{1}{2}\frac{q^2}{\epsilon A} g = W(q)}$$

** — = —

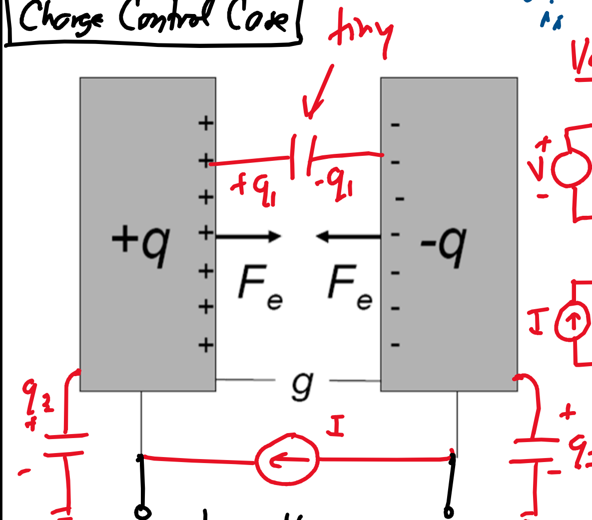
Case: $R_S \ll R_L$ then $N_o = N_i$

Case $R_S \gg R_L$ then $N_o \to 0$ but $i_o = i_i$

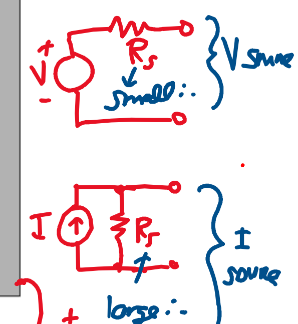
$N_o = \frac{R_L}{R_S + R_L} N_i$



Charge Control Case



V or I in?



From $dW = Vdq + F_e dg$

⇒ Force is given by:

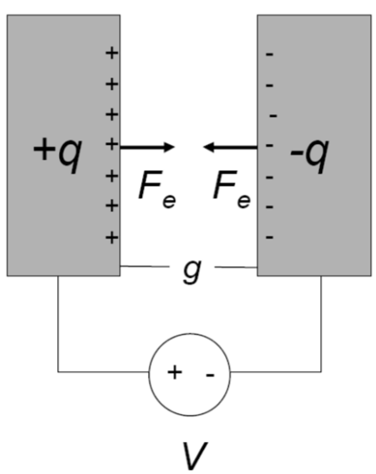
$$F_e = \frac{\partial W(q,g)}{\partial g}\Big|_{q=const.} = \frac{\partial}{\partial g}\left(\frac{1}{2}\frac{q^2}{\epsilon A}g\right)$$

$$\therefore \boxed{F_e = \frac{1}{2}\frac{q^2}{\epsilon A}}$$ ⇒ independent of gap spacing!

⇒ voltage takes the form:

$$V = \frac{\partial W(q,g)}{\partial q}\Big|_{g=const.} = \frac{\partial}{\partial q}\left(\frac{1}{2}\frac{q^2}{\epsilon A}g\right) = \frac{qg}{\epsilon A} \Rightarrow \boxed{V = \frac{q}{C}} \checkmark$$

Voltage Control



Want to write $F_e = f(V)$

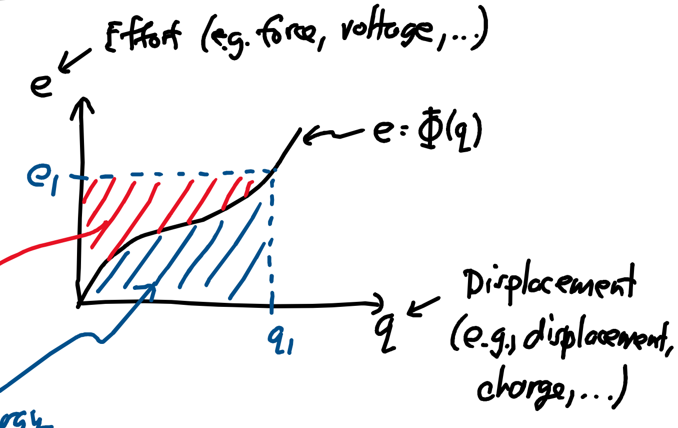
We know this: $dW = Vdq + F_e dg$

$\rightarrow W = W(q,g)$

Need: $W'(V,g)$ ← replace charge q w/ V

Can get this using a Legendre transformation.

Energy & Co-Energy



Energy

$$W(q_1) = \int_0^{q_1} e\,dq = \int_0^{q_1}\Phi(q)\,dq$$

Co-Energy:

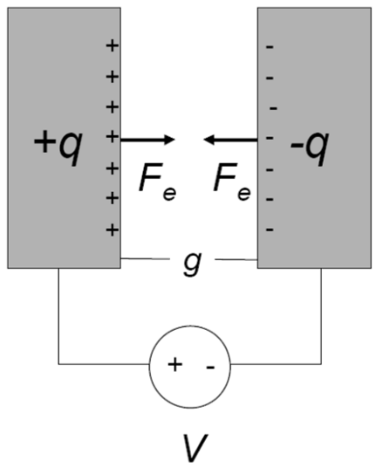
$$W'(e_1) = \int_0^{e_1} q\,de = \int_0^{e_1}\Phi^{-1}(e)\,de$$

For a linear system, these will be equal.

Can define co-energy as:

$$\underset{\text{co-energy}}{W'(e)} = eq - \underset{\text{energy}}{W(q)}$$ (from the plot) *

$\boxed{\text{Co-Energy Formulation for Voltage Control}}$



* ← $W'(V,g) = Vq - W(q,g)$

Differentially, this becomes

$$dW'(V,g) = (q\,dV + \cancel{V\,dq}) - dW(q,g)$$

$$\left[dW(q,g) = \cancel{V\,dq} + F_e\,dg\right]$$

$$\boxed{dW'(V,g) = q\,dV - F_e\,dg}$$ ← Working Co-Energy Expression

Find co-energy in terms of voltage V:

$$W' = \int_0^V q(q,V')\,dV' = \int_0^V \left(\frac{\epsilon A}{g}\right) V'\,dV'$$

$$= \frac{1}{2}\left(\frac{\epsilon A}{g}\right)V^2 = \underline{\frac{1}{2}CV^2} \checkmark \text{ (as expected)}$$

Electrostatic (or Voltage-Controlled) Force:

$$F_e = -\left.\frac{\partial W'(V,g)}{\partial g}\right|_{V=\text{const.}}$$

$$= +\frac{1}{2}\left(\frac{\epsilon A}{g^2}\right)V^2 = \boxed{\frac{1}{2}\frac{C}{g}V^2 = F_e}$$

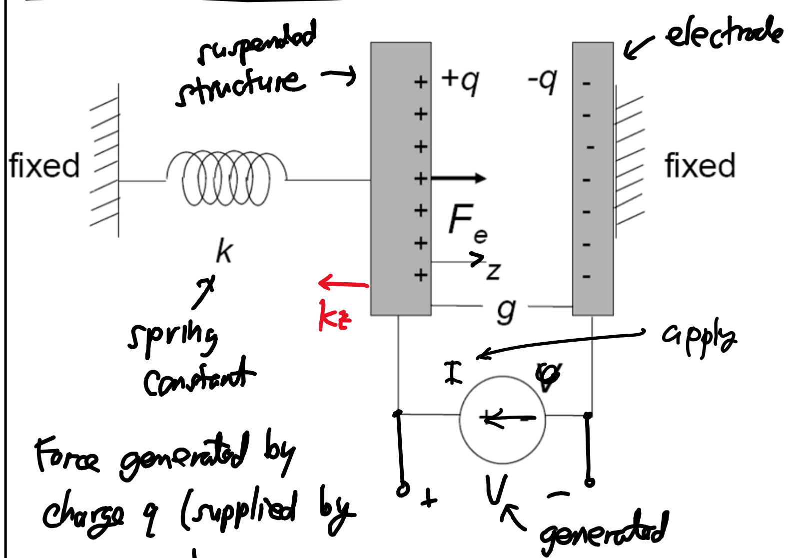
depends on gap!

Charge:

$$q = \left.\frac{\partial W'(V,g)}{\partial V}\right|_{g=\text{const.}} = \frac{\epsilon A}{g}V = CV \checkmark$$

(as expected)

Charge-Control of a Spring-Suspended C



Force generated by charge q (supplied by current I):

$$F_e = \frac{\partial W(q,g)}{\partial g}\bigg|_q = \frac{q^2}{2\epsilon A}$$

Restoring force of spring: $F_{spring} = kz = F_e$ (equilibrium)

The gap:

$$g = g_0 - z = g_0 - \frac{F_e}{k} = \boxed{g_0 - \frac{1}{2}\frac{q^2}{\epsilon A}\frac{1}{k} = g}$$

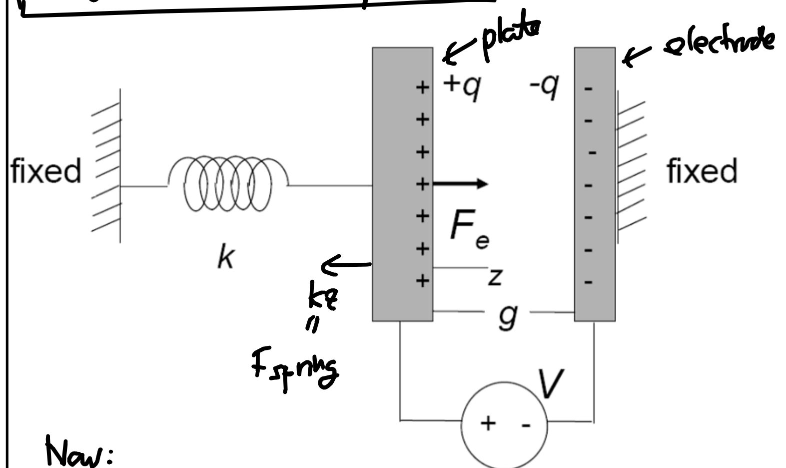
g_0: initial gap

$q\uparrow$ can drive $g \to 0$ in a continuous fashion

$$V = \frac{q}{C} = \frac{q}{\epsilon A} g = \boxed{\frac{q}{\epsilon A}\left(g_0 - \frac{1}{2}\frac{q^2}{\epsilon A}\frac{1}{k}\right) = V}$$

← $V\downarrow$ as $g\downarrow$

Voltage-Control of a Suspended C



Now:

$$F_e = \frac{\partial W'(V,g)}{\partial g}\bigg|_{V=const.} \rightarrow F_e = \frac{1}{2}\frac{\epsilon A}{g^2}V^2$$

And the gap:

$$g = g_0 - z = g_0 - \frac{F_e}{k} = \boxed{g_0 - \frac{1}{2}\frac{\epsilon A V^2}{g^2 k} = g}$$

g_0: initial gap spacing

g shows up on both sides!

If $V\uparrow \rightarrow g\downarrow \rightarrow F_e\uparrow$

(+) Feedback!

If loop gain > 1, then this will go unstable!

→ plate will collapse to electrode (shorts → destroys!)

Charge: (for a stable gap)

$$q = \left.\frac{\partial W'(V,g)}{\partial V}\right|_g = CV \checkmark \quad \text{(as expected)}$$

$\boxed{\text{Stability Analysis}}$

⇒ determine under what conditions voltage control will collapse the plates

$$F_{net} = F_e - F_{spring} = \underbrace{\frac{\epsilon A V^2}{2g^2}}_{F_e} - \underbrace{k(g_o - g)}_{F_{spring}}$$

What happens when g changes by dg?
↳ get an increment of net attractive force F_{net}

$$dF_{net} = \frac{\partial F_{net}}{\partial g} dg = \left[-\frac{\epsilon A V^2}{g^3} + k\right] dg$$

If $g\downarrow \rightarrow dg = (-)$, then for stability need $F_{net}\downarrow \rightarrow dF_{net} = (-)$

∴ this must be (+)! → otherwise, plate collapses

Thus:

$$\boxed{k > \frac{\epsilon A V^2}{g^3}}$$

(for a stable uncollapsed system)