Lecture 20: Electrical Stiffness

- **Announcements:**
- Project Slide Set #1 due Friday, April 12
- HW#6 online and due Tuesday, 4/23, at 9 a.m.
- I will be traveling next week; should be back Thursday morning, 4/18 but we'll see if my plane lands in enough time for me to do the lecture
  - Tuesday, 4/16, will be a video lecture
  - Thursday, 4/18 lecture: we'll see
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- Reading: Senturia, Chpt. 5, Chpt. 6
- Lecture Topics:
  - Energy Conserving Transducers
    - Charge Control
    - Voltage Control
  - Parallel-Plate Capacitive Transducers
    - Linearizing Capacitive Actuators
    - Electrical Stiffness
  - Electrostatic Comb-Drive
    - 1st Order Analysis
    - 2nd Order Analysis
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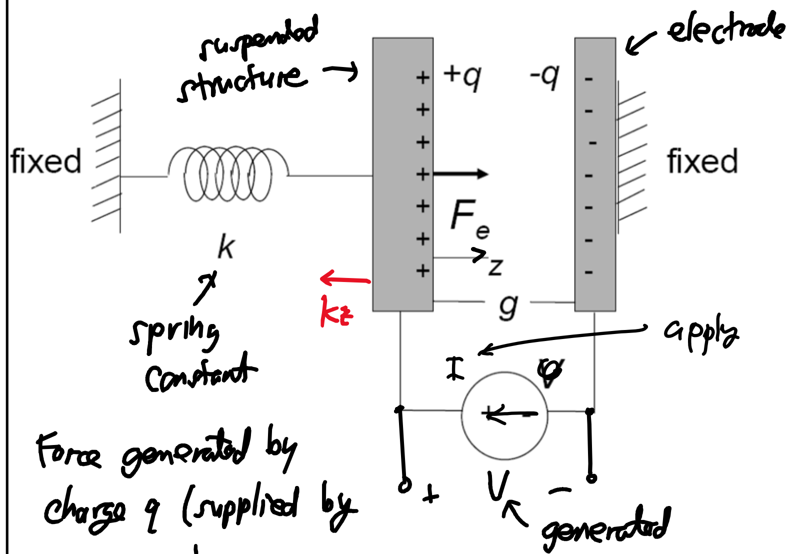
- **Last Time:**
- Going through pull in
- Now, continue with this …

Charge-Control of a Spring-Suspended C



Force generated by charge q (supplied by current I):

$$F_e = \frac{\partial W(q,g)}{\partial g}\bigg|_q = \frac{q^2}{2\epsilon A}$$

Restoring force of spring: $F_{spring} = kz = F_e$ ← equilibrium

The gap:

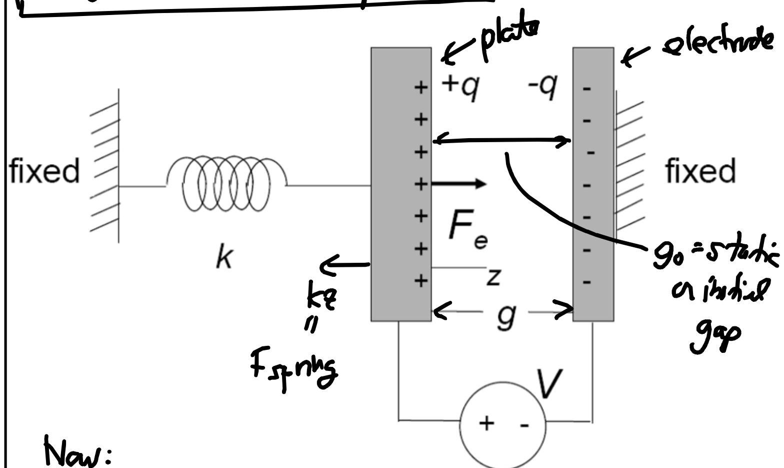
$$g = g_o - z = g_o - \frac{F_e}{k} = \boxed{g_o - \frac{1}{2}\frac{q^2}{\epsilon A}\frac{1}{k} = g}$$

↑ initial gap

↳ $q\uparrow$ can drive $g \to 0$ in a continuous fashion

$$V = \frac{q}{C} = \frac{q}{\epsilon A} g = \boxed{\frac{q}{\epsilon A}\left(g_o - \frac{1}{2}\frac{q^2}{\epsilon A}\frac{1}{k}\right) = V}$$ ← $V\downarrow$ as $g\downarrow$

Voltage-Control of a Suspended C



Now:

$$F_e = \left.\frac{\partial W'(V,g)}{\partial g}\right|_{V=const.} \rightarrow F_e = \frac{1}{2}\frac{\epsilon A}{g^2}V^2$$

And the gap:

$$g = g_0 - z = g_0 - \frac{F_e}{k} = \boxed{g_0 - \frac{1}{2}\frac{\epsilon A}{g^2}\frac{V^2}{k} = g}$$

g_0: initial gap spacing

g shows up on both sides!

If $V\uparrow \rightarrow g\downarrow \rightarrow F_e\uparrow$

(+) Feedback!

If loop gain > 1, then this will go unstable!

plate will collapse to electrode (shorts → destroys!)

Charge: (for a stable gap)

$$q = \left.\frac{\partial W'(V,g)}{\partial V}\right|_g = CV \checkmark \quad \text{(as expected)}$$

Stability Analysis

⇒ determine under what conditions voltage-control will collapse the plates

$$F_{net} = F_e - F_{spring} = \underbrace{\frac{\epsilon A V^2}{2g^2}}_{F_e} - \underbrace{k(g_0 - g)}_{F_{spring}}$$

What happens when g changes by dg?

↳ get an increment of net attractive force F_{net}

$$dF_{net} = \frac{\partial F_{net}}{\partial g}dg = \left[-\frac{\epsilon A V^2}{g^3} + k\right]dg$$

If $g\downarrow \rightarrow dg = (-)$, then for stability need $F_{net}\downarrow \rightarrow dF_{net} = (-)$

∴ this must be (+)! → otherwise, plate collapses

Thus:

$$\boxed{k > \frac{\epsilon A V^2}{g^3}}$$

(for a stable uncollapsed system)

Pull-in Voltage & Pull-in Gap

$V_{PI} \triangleq$ voltage @ which plates collapse

$g_{PI} \triangleq$ gap @ " " "

The plate goes unstable when:

$$k = \frac{\epsilon A V_{PI}^2}{g_{PI}^3} \qquad (1)$$

$$F_{net} = 0 = \underbrace{\frac{\epsilon A V_{PI}^2}{2 g_{PI}^2}}_{F_e} - \underbrace{k(g_o - g_{PI})}_{F_{spring}} \qquad (2)$$

Substitute (1) into (2):

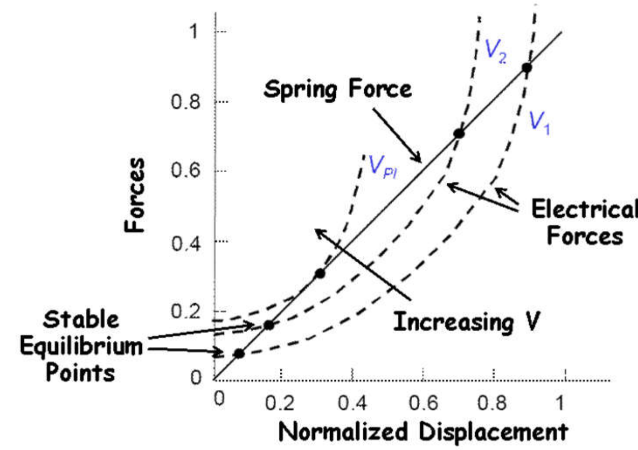
$$0 = \frac{\cancel{\epsilon A V_{PI}^2}}{2\cancel{g_{PI}^2}} - \frac{\cancel{\epsilon A V_{PI}^2}}{g_{PI}^{\cancel{3}}}(g_o - g_{PI})$$

$$\frac{g_o - g_{PI}}{g_{PI}} = \frac{1}{2} \rightarrow g_o = \frac{3}{2} g_{PI}$$

$$\therefore \boxed{g_{PI} = \frac{2}{3} g_o}$$

When the gap is driven by a voltage to (2/3) the initial gap → collapse!

$$V_{PI} = \sqrt{\frac{k g_{PI}^3}{\epsilon A}} \rightarrow \boxed{V_{PI} = \sqrt{\frac{8}{27}\frac{k g_o^3}{\epsilon A}}}$$



Advantages of Electrostatic Actuators:

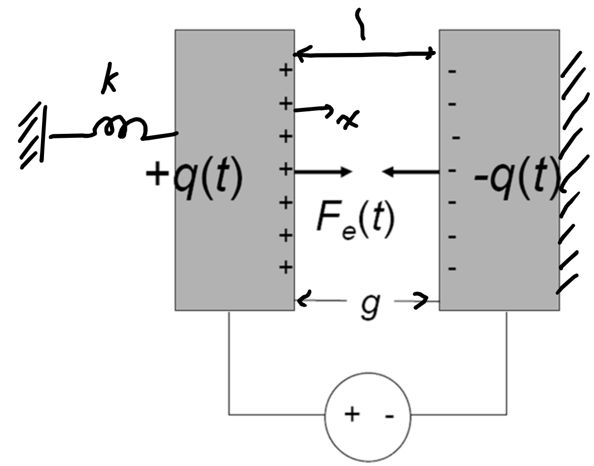
- Easy to manufacture in micromachining processes, since conductors and air gaps are all that's needed → low cost!
- Energy conserving → only parasitic energy loss through I^2R losses in conductors and interconnects
- Variety of geometries available that allow tailoring of the relationships between voltage, force, and displacement
- Electrostatic forces can become very large when dimensions shrink → electrostatics scales well!
- Same capacitive structures can be used for both drive and sense of velocity or displacement
- Simplicity of transducer greatly reduces mechanical energy losses, allowing the highest Q's for resonant structures

Disadvantages of Electrostatic Actuators:

- Nonlinear voltage-to-force transfer function
- Relatively weak compared with other transducers (e.g., piezoelectric), but things get better as dimensions scale
- Go through variable naming convention in slide 21 of Lecture Module 12

Linearizing the Voltage-to-Force Transfer Function

g_o: initial gap spacing



$$v(t) = \underbrace{V_P}_{\text{DC Bias}} + \underbrace{v_i(t)}_{\text{AC signal}}$$

Large

small

$$F_e(t) = \frac{\partial W'}{\partial x} = \frac{\partial}{\partial x}\left[\frac{1}{2}C\,[v(t)]^2\right]$$

$$= \frac{1}{2}\frac{\partial C}{\partial x}[v(t)]^2 = \frac{1}{2}\frac{\partial C}{\partial x}[V_P + v_i(t)]^2$$

$$= \frac{1}{2}\frac{\partial C}{\partial x}\left[V_P^2 + 2V_P v_i(t) + \cancel{[v_i(t)]^2}\right]$$

$$(V_P \gg v_i(t)) \Rightarrow F_e(t) = \underbrace{\frac{1}{2}V_P^2\frac{\partial C}{\partial x}}_{\text{DC offset}} + \underbrace{V_P\frac{\partial C}{\partial x}v_i(t)}_{\text{AC drive signal}}$$

$$C_0 = \frac{\epsilon A}{g_0} \rightarrow C(x) = \frac{\epsilon A}{g_0 - x} = C_0\left(1 - \frac{x}{g_0}\right)^{-1}$$

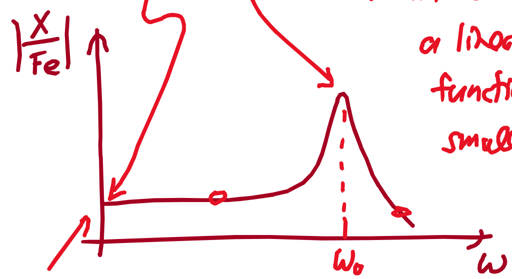
$$[x \ll g_0] \Rightarrow \quad \approx C_0\left(1 + \frac{x}{g_0}\right)$$

$$\therefore \frac{\partial C}{\partial x} = \frac{C_0}{g_0} = \frac{\epsilon A}{g_0^2}$$

$$\Rightarrow F_e(t) = \underbrace{\frac{1}{2}\frac{C_0}{g_0}V_P^2}_{\text{DC Offset}} + V_P \frac{C_0}{g_0} v_i(t)$$

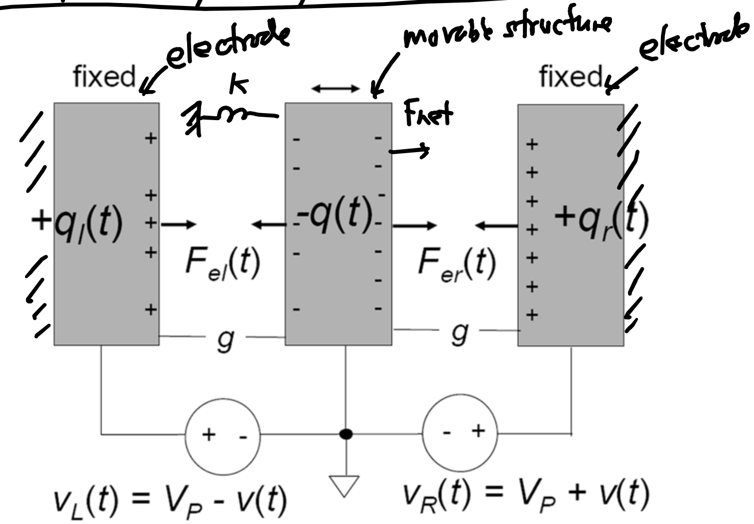
• constant fn small amplitudes

∴ this effectively is a linear transfer function (fn small amplitudes)



very small response → But... still must worry about V_{PI}!

Can Cancel the DC Offset & Squared Term Via Differential Symmetry



$$F_{net}(t) = F_{er}(t) - F_{el}(t)$$

$$= \frac{1}{2}\frac{\partial C}{\partial x}\left\{[v_R(t)]^2 - [v_L(t)]^2\right\}$$

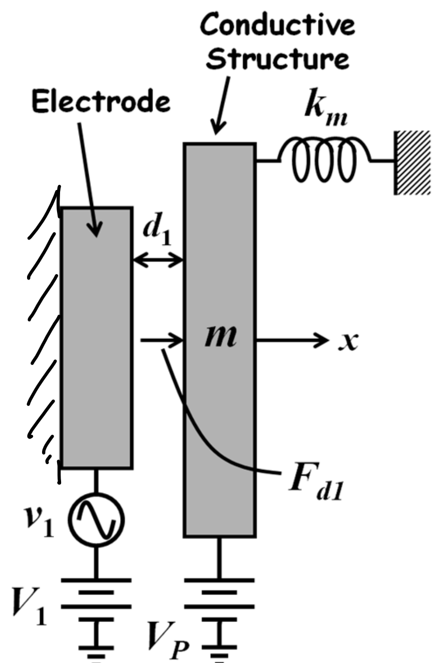
$$= \frac{1}{2}\frac{\partial C}{\partial x}\left\{\cancel{V_P^2} + 2V_P v(t) + \cancel{[v(t)]^2} - \left(\cancel{V_P^2} - 2V_P v(t) + \cancel{[v(t)]^2}\right)\right\}$$

cancelled out the nonlinear voltage term!

$$\boxed{\begin{aligned} \therefore F_{net}(t) &= 2V_P \frac{\partial C}{\partial x} v(t) \\ &= 2V_P \frac{C_0}{g_0} v(t) \end{aligned}}$$

quite linear w/ $v(t)$

Nonlinearity Still Effects v_s



More Complete Expression for $\frac{\partial C}{\partial x}$

$$C_1(x) = \frac{\epsilon A}{d_1+x} = C_{o1}\left(1+\frac{x}{d_1}\right)^{-1} \rightarrow \frac{\partial C_1}{\partial x} = -\frac{C_{o1}}{d_1}\left(1+\frac{x}{d_1}\right)^{-2}$$

[Expand into Taylor Series]

$$\frac{\partial C_1}{\partial x} = -\frac{C_{o1}}{d_1}\left(1 + A_1 x + A_2 x^2 + A_3 x^3 + \cdots\right)$$

where $A_1 = -\frac{2}{d_1}$, $A_2 = \frac{3}{d_1^2}$, $A_3 = -\frac{4}{d_1^3}$, ...

$$F_{d1} = \frac{1}{2}\frac{\partial C_1}{\partial x}(V_P - V_1 - v_1)^2 = \frac{1}{2}\frac{\partial C}{\partial x}(V_{P1} - v_1)^2$$

$V_{P1} = V_P - V_1$ $\frac{1}{2}(1+\cos 2\omega_o t)$ → $2\omega_o$

" $\cos^2\omega_o t$

[small displacements: $x \ll d_1$]

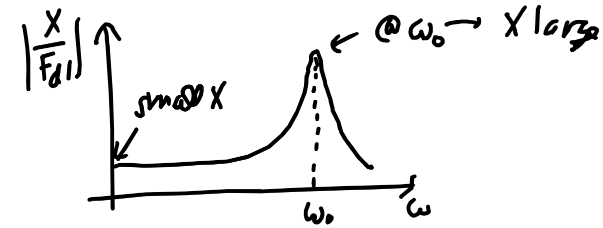
$$F_{d1} = \frac{1}{2}\left(-\frac{C_{o1}}{d_1}\right)(1 + A_1 x)\left(V_{P1}^2 - 2V_{P1}v_1 + v_1^2\right)$$

$$= \frac{1}{2}\left(-\frac{C_{o1}}{d_1}\right)\left\{V_{P1}^2 - 2V_{P1}v_1 + v_1^2 + A_1 V_{P1}^2 x - 2A_1 V_{P1} x v_1 + A_1 x v_1^2\right\}$$

but small so ignore

⇒ Focus on the signal voltage: $v_1(t)$

⇒ @ resonance (for maximum response)



@ resonance:

$$x = \frac{QF_{d1}}{jk} = \frac{Q}{jk}\frac{\partial C}{\partial x}V_{P1}v_1$$

x is 90° phase shifted from F_{d1} @ resonance

$v_1 = |v_1|\cos\omega_o t \rightarrow x = |x|\sin\omega_o t$

90° phase-shifted

Force terms @ ω_o

$$F_{d1}\Big|_{\omega_o} = \underbrace{V_{P1}\frac{C_{o1}}{d_1}|v_i|\cos\omega_o t}_{\text{drive force term}} + \overbrace{V_{P1}^2 \frac{C_{o1}}{d_1^2}}^{k_e}|x|\sin\omega_o t$$

$k_e \rightarrow$ electrical stiffness

proportional to x!

drive force term

90° phase-shifted f/

$\therefore$ in phase w/ displacement!

$\therefore$ it's a stiffness!

Electrical Stiffness:

① A negative spring constant!

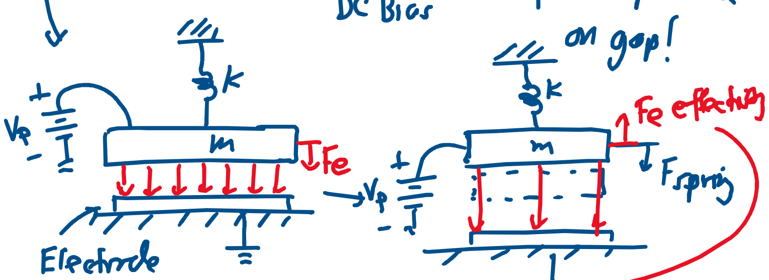
② Derives from V_P:

$$\boxed{k_e = V_{P1}^2\frac{C_{o1}}{d_1^2} = V_{P1}^2\frac{\epsilon A}{d_1^3}}$$

overlap area of the C

DC Bias

3rd power dependence on gap!



$\therefore$ negative stiffness!

$k_e \rightarrow$ can influence resonance freq. ω_o

$\omega_o \triangleq$ radian resonance freq. w/ no V_P applied (i.e., w/ $V_{P1} = 0V$)

$$\omega_o' = \sqrt{\frac{k_{tot}}{m}} = \sqrt{\frac{k_m - k_e}{m}}$$

k_m = mech. stiffness

$$= \sqrt{\frac{k}{m}}\left(1 - \frac{k_e}{k_m}\right)^{1/2}$$

$$\boxed{\omega_o' = \omega_o\left[1 - \frac{V_{P1}^2}{k_m}\frac{\epsilon A}{d_1^3}\right]^{1/2}}$$

now a fcn of dc bias V_{P1}!

(voltage-controllable!)