Lecture 22: Equivalent Circuits II

- Announcements:
- Project Slide Set #2 due Friday, April 19
- HW#6 online and due Tuesday, 4/23, at 9 a.m.
- Module 13 on Equivalent Circuits II online
- I am traveling, so this is a video lecture
  - I should be back Thursday morning, 4/18 but we'll see if my plane lands in enough time for me to do the lecture
  - If not, the lecture will be via video
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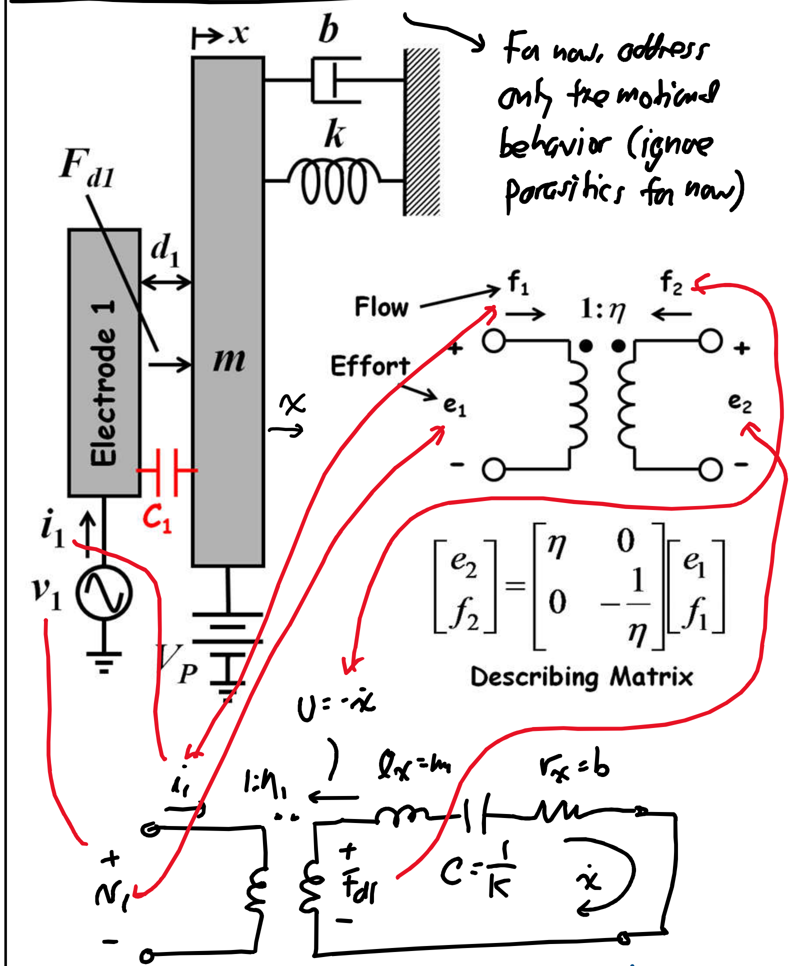
- Reading: Senturia, Chpt. 6, Chpt. 14
- Lecture Topics:
  - Input Modeling
    - Force-to-Velocity Equiv. Ckt.
    - Input Equivalent Ckt.
  - Current Modeling
    - Output Current Into Ground
    - Input Current
    - Complete Electrical-Port Equiv. Ckt.
  - Impedance & Transfer Functions
- ------------------------------------
- Last Time:
- In the process of developing an equivalent circuit for a capacitive-gap transduced mechanical structure
- Now, continue with this …

Input Electrical Equiv. Ckt.



$$\begin{bmatrix} e_2 \\ f_2 \end{bmatrix} = \begin{bmatrix} \eta & 0 \\ 0 & -\frac{1}{\eta} \end{bmatrix} \begin{bmatrix} e_1 \\ f_1 \end{bmatrix}$$

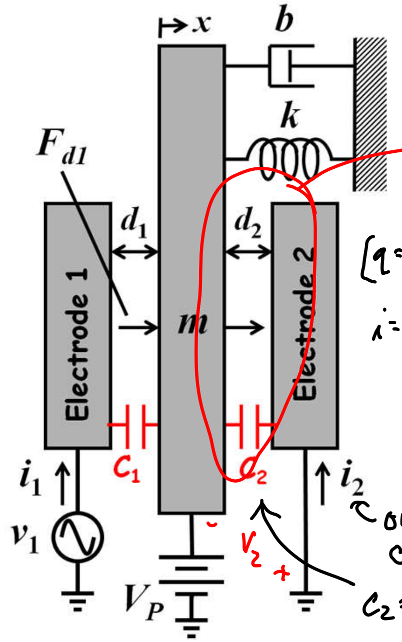
$$\underbrace{F_{d1}}_{e_2} \equiv \underbrace{-V_P \frac{\partial C_1}{\partial x}}_{\eta_1} \underbrace{v_1}_{e_1}$$

$$\boxed{\eta_1 = \left| V_P \frac{\partial C_1}{\partial x} \right|}$$

$\frac{\epsilon_o A}{d_1^2}$ ↳ gap $d \rightarrow \eta_1 \uparrow$

$V_P \uparrow \rightarrow \eta_1 \uparrow$

Output Current Into Ground



$[q = CV]$

$i = \frac{dq}{dt} = C\frac{dv}{dt} + V\frac{dC}{dt}$

$$i_2 = C_2(x,t)\cancel{\frac{dV_2(t)}{dt}} + V_2(t)\frac{dC_2(x,t)}{dt}$$

$$[V_2(t) = -V_P \neq f(t)] \Rightarrow i_2 = -V_P\frac{dC_2}{dt} = -V_P\frac{\partial C_2}{\partial x}\frac{\partial x}{\partial t}$$

In phasor form: $I_2(j\omega) = -V_P\frac{\partial C_2}{\partial x}(j\omega X)$

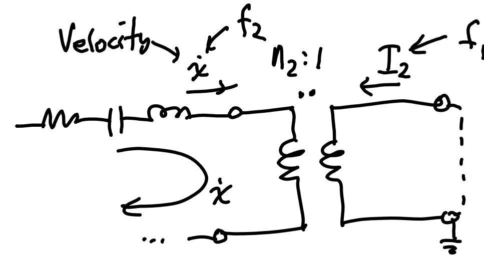
$$\boxed{I_2(j\omega) = -j\omega V_P\frac{\partial C_2}{\partial x}X}$$

output motional current

$$I_2(j\omega) = -j\omega V_P\frac{\partial C_2}{\partial x}X = -V_P\frac{\partial C}{\partial x}\dot{x}$$

velocity

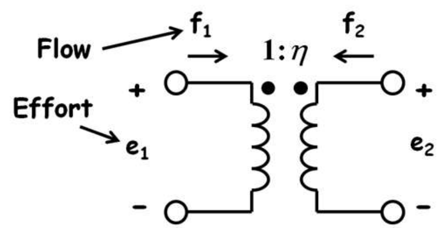
90° phase lag (+) (-) } $I_2 = (-)$ when $X = (+)$



$f_2 = -\frac{1}{\eta_2}f_1 \rightarrow f_1 = -\eta_2 f_2$

$[f_1 = I_2, f_2 = \dot{x}] \Rightarrow I_2 = -\eta_2\dot{x}$

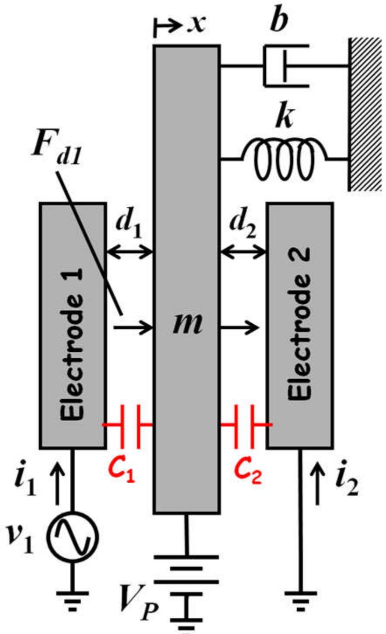
$\therefore \boxed{\eta_2 = \left|V_P\frac{\partial C_2}{\partial x}\right|}$



$$\begin{bmatrix}e_2\\f_2\end{bmatrix} = \begin{bmatrix}\eta & 0\\0 & -\frac{1}{\eta}\end{bmatrix}\begin{bmatrix}e_1\\f_1\end{bmatrix}$$

Describing Matrix

$\boxed{\text{Input Current Expression}}$ ⇒ then yields complete equivalent input ckt



Get $I_1(j\omega)$:

$$i_1(t) = C_1(x,t)\frac{dV_1(t)}{dt} + V_1(t)\frac{dC_1(x,t)}{dt}$$

$$[V_1(t) = \underset{\uparrow\, f(t)}{v_1} - V_P] \Rightarrow i_1 = C_1\frac{dv_1}{dt} + (v_1 - V_P)\frac{\partial C_1}{\partial x}\frac{\partial x}{\partial t}$$

magnitude of $v_1 = |v_1|\cos\omega t$

$$\therefore\ I_1(j\omega) = \underbrace{j\omega C_1|v_1|}_{\text{Feedthrough Current}} + \cancel{j\omega|v_1|\frac{\partial C_1}{\partial x}X} - \underbrace{j\omega V_P\frac{\partial C_1}{\partial x}X}_{\text{Motional Current due to mass motion}} \longrightarrow \star$$

@DC: $x = \frac{F_{d1}}{k} = -\frac{1}{k}V_P\frac{\partial C_1}{\partial x}v_1$

@ resonance: $x = \frac{QF_{d1}}{k} = -\underset{\substack{\uparrow \\ 90^\circ \text{ phase lag}}}{\frac{Q}{jk}}V_P\frac{\partial C_1}{\partial x}v_1 = X$

plug in to ★

Thus: (@ resonance $\to \omega_o$)

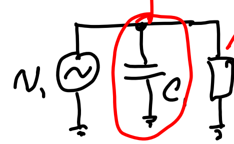
$$I_1(j\omega) = j\omega_o C_1|v_1| + j\omega_o\Big(\overbrace{V_P\frac{\partial C_1}{\partial x}}^{\eta_{e1}}\Big)^2\frac{Q}{jk}|v_1|$$

$$= j\omega_o C_1|v_1| + \omega_o\frac{Q}{k}\eta_{e1}^2|v_1|$$

90° phase-shifted from v_1 ↓ This is a capacitor in shunt w/ input!

In phase w/ v_1! ↑ This is an effective resistance @ ω_o seen looking into the electrode!

@ resonance

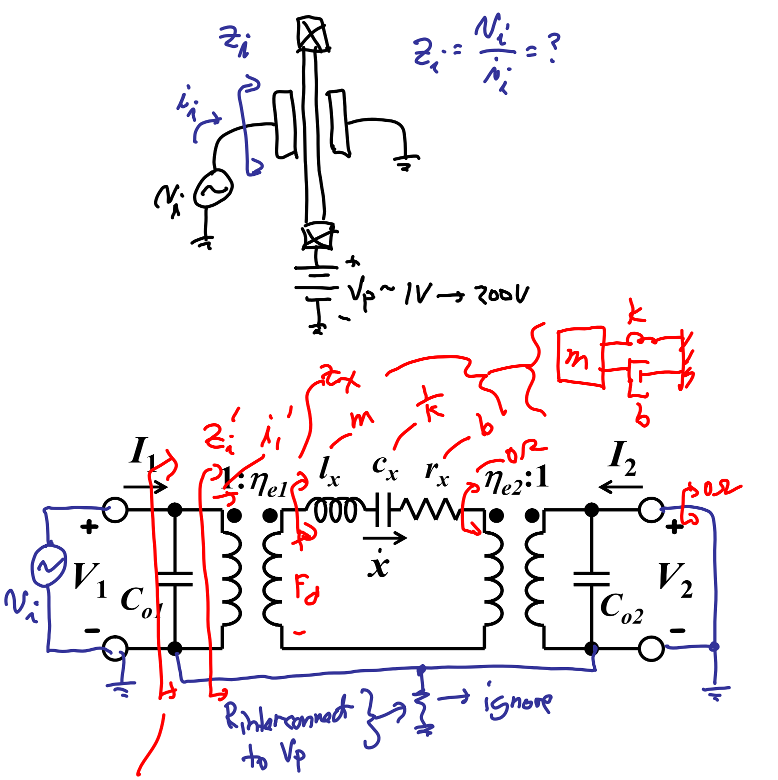


Motional Resistance:

$$R_{x1} = \frac{v_1}{I_1} = \frac{k}{\omega_o Q\eta_{e1}^2} = \frac{m\omega_o}{Q\eta_{e1}^2} = \boxed{\frac{b}{\eta_{e1}^2} = R_{x1}}$$

The equivalent ckt. better get this right!

Input Impedance Into Port 1



$z_i : = \frac{v_i}{i_i} = ?$

$$z_i = \frac{v_i}{i_i} = C_{o1} \| z_i'$$

$$\hookrightarrow z_i' = \frac{v_i}{i_i'} = \frac{e_1}{f_1}$$

$$\begin{bmatrix} e_2 \\ f_2 \end{bmatrix} = \begin{bmatrix} \eta & 0 \\ 0 & -\frac{1}{\eta} \end{bmatrix} \begin{bmatrix} e_1 \\ f_1 \end{bmatrix} \Rightarrow \begin{array}{l} e_2 = \eta e_1 \to e_1 = \frac{e_2}{\eta} \\ f_2 = -\frac{1}{\eta} f_1 \to f_1 = -\eta f_2 \end{array}$$

$$\frac{e_1}{f_1} = z_i' = \frac{e_2}{\eta}\left(\frac{1}{-\eta f_2}\right) = -\frac{1}{\eta^2}\frac{e_2}{f_2} \to \frac{v_i}{i_i'} = z_i' = \cancel{-}\frac{1}{\eta_{e1}^2}\frac{F_d}{(\cancel{-}\dot{x})}$$

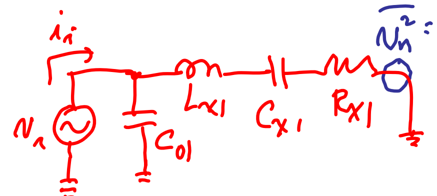
$$z_i' = \frac{1}{\eta_{e1}^2} Z_x$$

$$z_i' = \frac{1}{\eta_{e1}^2}\left(j\omega l_x + \frac{1}{j\omega c_x} + r_x\right)$$

$$= \underbrace{j\omega\left(\frac{l_x}{\eta_{e1}^2}\right)}_{L_{x1}} + \underbrace{\frac{1}{j\omega(\eta_{e1}^2 c_x)}}_{C_{x1}} + \underbrace{\frac{r_x}{\eta_{e1}^2}}_{R_{x1}}$$

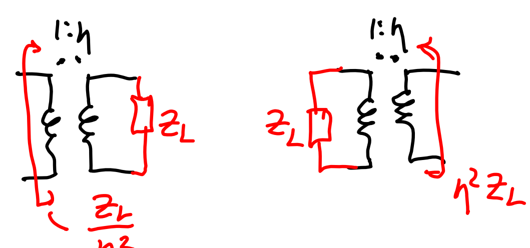
← models noise!

$\overline{v_n^2} = 4kTR_x$

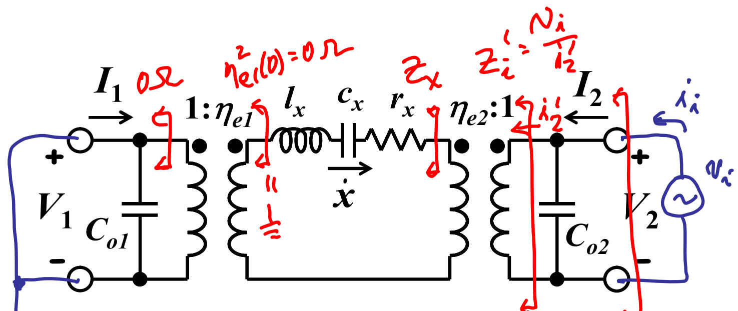


Purely Electrical Equiv. Ckt.

Xformer Inspection Analysis



Input Impedance Into Port 2

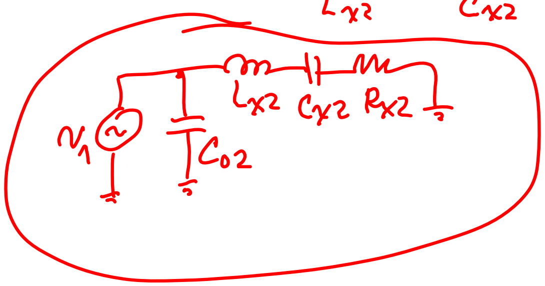


← Ground out port 1

$Z_i = \frac{v_i}{i_i} = ?$

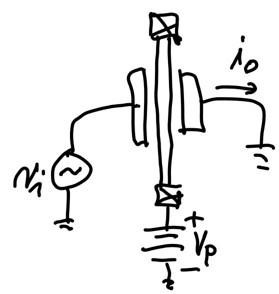
$= C_{o2} \| z_i'$

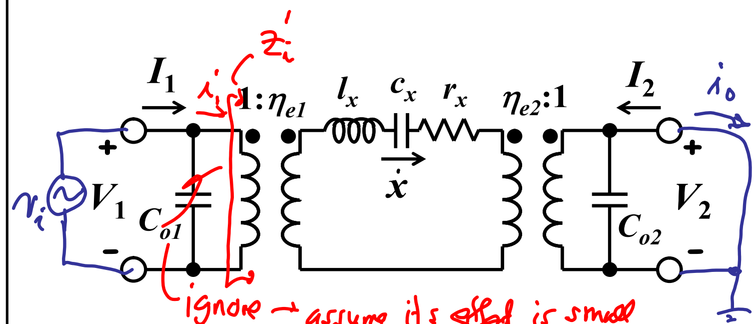
$$z_i' = \frac{v_i}{i_2'} = \frac{Z_x}{\eta_{e2}^2} = \underbrace{j\omega\left(\frac{l_x}{\eta_{e2}^2}\right)}_{L_{x2}} + \underbrace{\frac{1}{j\omega(\eta_{e2}^2 c_x)}}_{C_{x2}} + \underbrace{\frac{r_x}{\eta_{e2}^2}}_{R_{x2}}$$



Port 1-to-Port 2 Transconductance

$G = \frac{i_o}{v_i} = ?$





ignore → assume its effect is small compared to motional parameters (not always the case)

$$\left.\begin{aligned}\dot{x} &= \frac{1}{\eta_{e1}} i_1 \\ i_o &= \eta_{e2}\dot{x}\end{aligned}\right\} \quad i_o = \frac{\eta_{e2}}{\eta_{e1}} i_1 = \frac{\eta_{e2}}{\eta_{e1}}\left(\frac{v_i}{z_i'}\right)$$

$$i_o = \frac{\eta_{e2}}{\eta_{e1}}\left[\eta_{e1}^2 \frac{1}{j\omega l_x + \frac{1}{j\omega c_x} + r_x}\right] v_i$$

$$\therefore \frac{i_o}{v_i}(j\omega) = \frac{\eta_{e1}\eta_{e2}}{j\omega l_x + \frac{1}{j\omega c_x} + r_x}$$

$$\boxed{\frac{i_o}{v_i}(j\omega) = \left[j\omega L_{x12} + \frac{1}{j\omega C_{x12}} + R_{x12}\right]^{-1}}$$

where $L_{x12} = \frac{l_x}{\eta_{e1}\eta_{e2}}$, $C_{x12} = \eta_{e1}\eta_{e2}c_x$, $R_{x12} = \frac{r_x}{\eta_{e1}\eta_{e2}}$

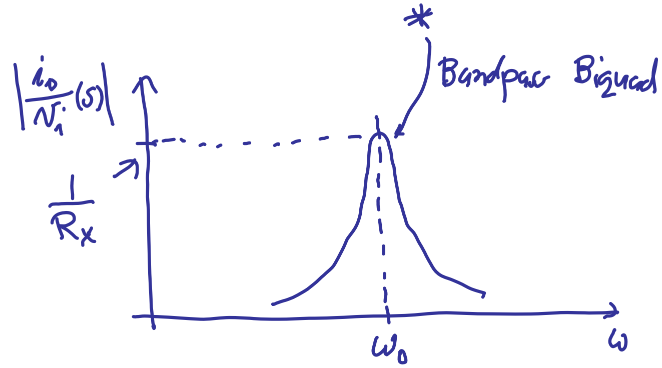
→ separate into magnitude & freq. response components

$$\frac{i_o}{v_i}(s) = \frac{1}{sL_x + \frac{1}{sC_x} + R_x} = \frac{s\left(\frac{1}{L_x}\right)}{s^2 + \frac{1}{L_xC_x} + s\left(\frac{R_x}{L_x}\right)}$$

$$\left(\frac{1}{L_xC_x} = \omega_0^2,\quad Q = \frac{\omega_0 L_x}{R_x} \rightarrow \frac{R_x}{L_x} = \frac{\omega_0}{Q}\right)$$

$$\frac{i_o}{v_i}(s) = \underbrace{\frac{1}{R_x}}_{\text{Gain Term}}\ \underbrace{\frac{s\left(\frac{\omega_0}{Q}\right)}{s^2 + s\left(\frac{\omega_0}{Q}\right) + \omega_0^2}}_{\text{Frequency Shaping Term}} = \underbrace{\frac{1}{R_x}}_{\text{resonance magnitude}} H(s)$$

*



To analyze mechanical ckt. problem:

1. Just solve the ckt. @ resonance → easy, since L & C cancel
2. Then multiply the result by $H(s)$!

- Now, go through slide 21 in Module 13
- Then, start gyroscopes in Module 15