Lecture Module 10: Resonance Frequency

- Reading: Senturia, Chpt. 10: §10.5, Chpt. 19
- Lecture Topics:
  - Estimating Resonance Frequency
  - Lumped Mass-Spring Approximation
  - ADXL-50 Resonance Frequency
  - Distributed Mass & Stiffness
  - Folded-Beam Resonator
Estimating Resonance Frequency

Clamped-Clamped Beam \( \mu \)Resonator

Resonator Beam

Electrode

Sinusoidal Excitation

Voltage-to-Force Capacitive Transducer

\[
v_i = V_i \cos(\omega_o t) \quad \Rightarrow \quad f_i = F_i \cos(\omega_o t)
\]

- \( \omega \neq \omega_o \): small amplitude
- \( \omega = \omega_o \): maximum amplitude \( \rightarrow \) beam reaches its maximum potential and kinetic energies
Estimating Resonance Frequency

* Assume simple harmonic motion:

\[ x(t) = x_0 \cos(\omega t) \]

* Potential Energy:

\[ W(t) = \frac{1}{2} k x^2(t) = \frac{1}{2} k x_0^2 \cos^2(\omega t) \]

* Kinetic Energy:

\[ K(t) = \frac{1}{2} M \dot{x}^2(t) = \frac{1}{2} M x_0^2 \omega^2 \sin^2(\omega t) \]

Estimating Resonance Frequency (cont)

* Energy must be conserved:

\[ \text{Potential Energy} + \text{Kinetic Energy} = \text{Total Energy} \]

\[ \text{Must be true at every point on the mechanical structure} \]

\[ W_{\text{max}} = \frac{1}{2} k x_0^2 = K_{\text{max}} = \frac{1}{2} M \omega^2 x_0^2 \]

* Solving, we obtain for resonance frequency:

\[ \omega = \sqrt{\frac{k}{M}} \]
Example: ADXL-50

- The proof mass of the ADXL-50 is many times larger than the effective mass of its suspension beams.
  - Can ignore the mass of the suspension beams (which greatly simplifies the analysis).
- Suspension Beam: \( L = 260 \, \mu m, \ h = 2.3 \, \mu m, \ W = 2 \, \mu m \)

Lumped Spring-Mass Approximation

- Mass is dominated by the proof mass:
  - 60% of mass from sense fingers
  - Mass = \( M = 162 \, \text{ng} \) (nano-grams)
- Suspension: four tensioned beams
  - Include both bending and stretching terms [A.P. Pisano, BSAC Inertial Sensor Short Courses, 1995-1998]
ADXL-50 Suspension Model

- Bending contribution:
  \[ k_b = \frac{1}{k_s} + \frac{1}{k_c} = 2 \left( \frac{L}{2E(Wh^3/12)} \right) = \frac{E^2}{3EWh^2} = 4.2 \mu \text{N} \mu \text{m} \]

- Stretching contribution:
  \[ k_s = \frac{k_s}{S} = \frac{L}{\alpha WH} = 1.14 \mu \text{N} \mu \text{m} \]

- Total spring constant: add bending to stretching (since they are in parallel)
  \[ k = 4(k_b + k_s) = 4(0.24 + 0.88) = 4.5 \mu \text{N} \mu \text{m} \]

\[ F_y = S \sin \theta = S(x/L) = \left( \frac{s}{L} \right) \sin \theta \]

ADXL-50 Resonance Frequency

- Using a lumped mass-spring approximation:
  \[ f = \frac{1}{2\pi} \sqrt{\frac{k}{M}} = \frac{1}{2\pi} \sqrt{\frac{4.48 \text{N/m}}{162 \times 10^{-12} \text{kg}}} = 26.5 \text{kHz} \]

- On the ADXL-50 Data Sheet: \( f_o = 24 \text{ kHz} \)
  - Why the 10% difference?
  - Well, it's approximate ... plus ...
  - Above analysis does not include the frequency-pulling effect of the DC bias voltage across the plate sense fingers and stationary sense fingers ... something we'll cover later on ...
Distributed Mechanical Structures

- Vibrating structure displacement function:
  \[ y(x, t) = \hat{y}(x) \cos(\omega t) \]

  Maximum displacement function (i.e., mode shape function)
  Seen when velocity \( \dot{y}(x, t) = 0 \)

- Procedure for determining resonance frequency:
  1. Use the static displacement of the structure as a trial function and find the strain energy \( W_{\text{max}} \) at the point of maximum displacement (e.g., when \( t=0, \pi/\omega, \ldots \))
  2. Determine the maximum kinetic energy when the beam is at zero displacement (e.g., when it experiences its maximum velocity)
  3. Equate energies and solve for frequency

Maximum Kinetic Energy

- Displacement: \( y(x, t) = \hat{y}(x) \cos[\omega t] \)

- Velocity: \( v(x, t) = \frac{\partial y(x, t)}{\partial t} = -\omega \hat{y}(x) \sin[\omega t] \)

- At times \( t = \pi/(2\omega), 3\pi/(2\omega), \ldots \)

  \[ y(x, t) = 0 \]

  Velocity topographical mapping

  - The displacement of the structure is \( y(x, t) = 0 \)
  - The velocity is maximum and all of the energy in the structure is kinetic (since \( W=0 \)):
    \[ v(x, (2n+1)\pi/(2\omega)) = -\omega \hat{y}(x) \]
Maximum Kinetic Energy (cont)

- At times $t = \pi/(2\omega), 3\pi/(2\omega), ...$

  \[ y(x, t) = 0 \]

  Velocity: \[ v(x, (2n + 1)\pi/(2\omega)) = -\omega \hat{y}(x) \]

  \[ dK = \frac{1}{2} \cdot dm \cdot [v(x, t)]^2 \]

  \[ dm = \rho(Wh \cdot dx) \]

- Maximum kinetic energy:

  \[ K_{\text{max}} = \int_0^L \frac{1}{2} \rho Wh dx v^2(x, t') = \frac{1}{2} \rho Wh \omega^2 \hat{y}^2(x) dx \]

The Raleigh-Ritz Method

- Equate the maximum potential and maximum kinetic energies:

  \[ K_{\text{max}} = \int_0^L \frac{1}{2} \rho Wh \omega^2 \hat{y}^2(x) dx = W_{\text{max}} \]

- Rearranging yields for resonance frequency:

  \[ \omega = \sqrt{\frac{W_{\text{max}}}{\int_0^L \frac{1}{2} \rho Wh \hat{y}^2(x) dx}} \]

\( \omega \) = resonance frequency

\( W_{\text{max}} \) = maximum potential energy

\( \rho \) = density of the structural material

\( W \) = beam width

\( h \) = beam thickness

\( \hat{y}(x) \) = resonance mode shape
**Example: Folded-Beam Resonator**

- Derive an expression for the resonance frequency of the folded-beam structure at left.

Use Rayleigh-Ritz method.

\[ KE_{\text{max}} = PE_{\text{max}} \]

Kinetic Energy:

\[ KE_{\text{max}} = KE_s + KE_t + KE_b \]

- Shuttle truss beams

\[ = \frac{1}{2} M_s \omega_s^2 + \frac{1}{2} M_t \omega_t^2 + \frac{1}{2} \int M_b \omega_b^2 \text{d}M_b \]

- Folding truss w/ mass \( M_t \)

\[ h = \text{thickness} \]

**Get Kinetic Energies**

- Velocity of the shuttle: \( \omega_s \Sigma_s \)

Resonance Freq. \( \rightarrow \)
Maximum Displacement Amplitude

\[ KE_s = \frac{1}{2} M_s \omega_s^2 + \frac{1}{2} M_b \omega_b^2 M_b \]

- Velocity of the truss: \( \omega_t \Sigma_t \)

\[ KE_t = \frac{1}{2} \left( \frac{1}{2} M_b \omega_b^2 \right) M_t = \frac{1}{2} M_b \omega_b^2 M_t \]

- Velocity of the beam segments:

- For segment AB:

\[ \beta(y) = \frac{F_x}{4 \pi E_t \left( 3y^2 - 2y^2 \right)} \quad 0 \leq y \leq L \]

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Folded-Beam Suspension

Comb-Driven Folded Beam Actuator

Get Kinetic Energies (cont)

Anchor

Shuttle w/ mass $M_1$

Folding truss w/ mass $M_1/12$

Static mass of beam $M(AB) \frac{d}{dx} \int_0^L \left[ 3 \left( \frac{x}{L} \right)^2 - 2 \left( \frac{x}{L} \right)^3 \right] dx$

$K_{E(AB)}^{(AB)} = \frac{13}{280} X_0^2 \omega_0^2 M(AB)$

Plugging in to express $K_{E_b}^{(AB)}$:

$K_{E_b}^{(AB)} = \frac{1}{2} \int_0^L \frac{X_0^2 \omega_0^2 M(AB)}{4} \left[ 3 \left( \frac{x}{L} \right)^2 - 2 \left( \frac{x}{L} \right)^3 \right] dM^{(AB)}$

Which yields for velocity:

$u_b(y) \bigg|_{(AB)} = \frac{X_0}{2} \left[ 3 \left( \frac{y}{L} \right)^2 - 2 \left( \frac{y}{L} \right)^3 \right] \omega_0$

Substituting into (1):

$\ddot{x}_b(y) = \frac{F_x}{4 \eta E_b} \left( 2 L y^2 - 2 y^3 \right)$, $0 \leq y \leq L$

$\ddot{x}_b(0) = 0 \checkmark$

$\ddot{x}_b(L) = \frac{F_x}{4 \eta E_b} L^3 \checkmark$

$K_{E_h} = k_2 \frac{12 \eta E_b}{L^2} \checkmark$

$h = \text{thickness}$
Get Kinetic Energies (cont)

For segment CD:

\[ \omega_0^2 \frac{d^2}{\theta} \left|_{(CD)} \right. = X_0 \left[ 1 - \frac{3}{2} \left( \frac{h}{L} \right)^2 + \left( \frac{h}{L} \right)^4 \right] \omega_0 \]

Thus:

\[ KE_{(CD)} = \frac{X_0^2 \omega_0^2 M_s}{2L} \int_0^L \left[ 1 - \frac{3}{2} \left( \frac{h}{L} \right)^2 + \left( \frac{h}{L} \right)^4 \right] dh \]

Static mass of beam [CD]

Let \( M_b \) be total mass of the 8 beams.

Then:

\[ M_{[AB]} = M_{[CD]} = \frac{1}{8} M_b \]

Thus:

\[ KE_b = \frac{4}{3} KE_{(AB)} + \frac{4}{3} KE_{(CD)} = \frac{6}{35} X_0^2 \omega_0^2 M_b \]

and

\[ KE_{max} = \frac{X_0^2 \omega_0^2}{2} M_s + \frac{1}{8} M_b + \frac{6}{35} M_b \]

Get Potential Energy & Frequency

PE\(_{max}\) is simply the work done to achieve maximum deflection:

\[ PE_{max} = \frac{1}{2} k_c X_0^2 \]

Thus, using Raleigh–Ritz:

\[ KE_{max} = PE_{max} \]

\[ X_0^2 \omega_0^2 \left[ \frac{1}{2} M_s + \frac{1}{8} M_b + \frac{6}{35} M_b \right] = \frac{1}{2} k_c X_0^2 \]

Resonance Frequency of a Folded-Beam Suspended Shuttle

\[ \omega_0 = \sqrt{\frac{k_c}{M_{eq}}} \]

Where \( M_{eq} = M_s + \frac{1}{8} M_b + \frac{12}{35} M_b \)
Brute Force Methods for Resonance Frequency Determination

Basic Concept: Scaling Guitar Strings

Guitar String

Vibrating “A” String (110 Hz)

Freq. Equation:

\[ f_0 = \frac{1}{2\pi} \sqrt{\frac{k_r}{m_r}} \]

Mechanical Resonator

Performance:

- \[ L = 40.8 \mu \text{m} \]
- \[ m_r \sim 10^{-13} \text{ kg} \]
- \[ W_r = 8 \mu \text{m}, h_r = 2 \mu \text{m} \]
- \[ d = 1000 \text{Å}, V_p = 5 \text{V} \]
- \[ \text{Press.} = 70 \text{mTorr} \]

\[ f_r = 8.5 \text{MHz} \]

\[ Q_{\text{vac}} = 8,000 \]

\[ Q_{\text{air}} \sim 50 \]
Anchor Losses

Fixed-Fixed Beam Resonator

Electrode

Gap

Anchor

Elastic Wave Radiation

Problem: direct anchoring to the substrate ⇒ anchor radiation into the substrate ⇒ lower Q

Free-Free Beam Resonator

Supporting Beams

Free-Free Beam

Solution: support at motionless nodal points ⇒ isolate resonator from anchors ⇒ less energy loss ⇒ higher Q

Q = 300 at 70MHz

Q = 15,000 at 92MHz

92 MHz Free-Free Beam μResonator

* Free-free beam mechanical resonator with non-intrusive supports ⇒ reduce anchor dissipation ⇒ higher Q

Design/Performance:

\[ L_p = 13.1 \mu m, W_p = 8 \mu m \]

\[ t = 2 \mu m, d = 1000 \AA \]

\[ V_p = 28-76V, W_p = 2.8 \mu m \]

\[ f_0 = 92.25 MHz \]

\[ Q = 7,450 @ 10 \text{mTorr} \]

[Wang, Yu, Nguyen 1998]
**Higher Order Modes for Higher Freq.**

**2nd Mode Free-Free Beam**
- Distinct Mode Shapes
- \( L = 20.3 \text{ \mu m} \)
- \( h = 2.1 \text{ \mu m} \)
- Electrodes
- Anchor
- Support Beam

**3rd Mode Free-Free Beam**

<table>
<thead>
<tr>
<th>Frequency [MHz]</th>
<th>Transmission [dB]</th>
<th>Phase [degree]</th>
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</thead>
<tbody>
<tr>
<td>101.31</td>
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<td>-180</td>
</tr>
<tr>
<td>101.34</td>
<td>-135</td>
<td>-135</td>
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<tr>
<td>101.37</td>
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<td>-90</td>
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<td>101.43</td>
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</tr>
<tr>
<td>101.46</td>
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<td>45</td>
</tr>
<tr>
<td>101.49</td>
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<tr>
<td>101.52</td>
<td>135</td>
<td>135</td>
</tr>
<tr>
<td>101.55</td>
<td>180</td>
<td>180</td>
</tr>
</tbody>
</table>

**Q = 11,500**

**Distinct Mode Shapes**

**Flexural-Mode Beam Wave Equation**

- Transverse Displacement: \( u \)
- Width: \( W \)
- \( z \)
- \( x \)
- \( L \)
- \( h \)

- Derive the wave equation for transverse vibration:
  - Dynamic Equilibrium Condition for Forces in the y-direction:
    \[
    F - (F + \frac{\partial F}{\partial x}dx) - \rho A \frac{\partial^2 u}{\partial t^2} = 0
    \]
    \[
    \text{neglect } \frac{\partial^2 F}{\partial x} dx \text{ term}
    \]
  - No moment equilibrium condition:
    \[
    -F dx + \frac{\partial M}{\partial x} dx \approx 0
    \]

- Combining (1) \& (2):
  \[
  \frac{\partial^2 M}{\partial x^2} dx = -\rho A \frac{\partial^2 u}{\partial t^2} \quad \text{and} \quad \frac{\partial^2 M}{\partial x^2} dx \approx 0
  \]
  \[
  \frac{\partial^2 u}{\partial t^2} = \frac{\rho A}{\frac{\partial^2 u}{\partial x^2}} \quad \text{and} \quad I_y = \frac{Wh^3}{12}
  \]
**Example: Free-Free Beam**

- Determine the resonance frequency of the beam
- Specify the lumped parameter mechanical equivalent circuit
- Transform to a lumped parameter electrical equivalent circuit
- Start with the flexural-mode beam equation:

\[
\frac{\partial^2 u}{\partial t^2} = \left( \frac{EI}{\rho A} \right) \frac{\partial^4 u}{\partial x^4}
\]

**Free-Free Beam Frequency**

- Substitute \( u = u_1 e^{j\omega t} \) into the wave equation:

\[
\frac{\partial^4 u}{\partial x^4} = \left( \frac{\omega^4 \rho A}{EI} \right) u
\]  

(1)

- This is a 4\(^{th}\) order differential equation with solution:

\[ u(x) = A \cosh kx + B \sinh kx + C \cos kx + D \sin kx \]  

(2)

**Boundary Conditions:**

\[
\begin{align*}
\text{At } x = 0 & : \frac{\partial^2 u}{\partial x^2} = 0, & M = 0 \text{ (Bending moment)} \\
\text{At } x = \ell & : \frac{\partial^3 u}{\partial x^3} = 0, & \frac{\partial M}{\partial x} = 0 \text{ (Shearing force)}
\end{align*}
\]
Free-Free Beam Frequency (cont)

- Applying B.C.'s, get $A=C$ and $B=D$, and

\[
\begin{bmatrix}
\cosh kl - \cos kl & \sinh kl - \sin kl \\
\sinh kl + \sin kl & \cosh kl - \cos kl
\end{bmatrix}
\begin{bmatrix}
A \\
B
\end{bmatrix} = 0 \quad (3)
\]

- Setting the determinant $= 0$ yields

\[
\cos kl = \frac{1}{\cosh kl}
\]

- Which has roots at

\[
k_1l = 4.730 \quad k_2l = 7.853 \quad k_3l = 10.996
\]

- Substituting (2) into (1) finally yields:

\[
k^4 = \frac{\rho A}{EI} \omega^2 \quad \Rightarrow \quad f_n = \frac{(k_nl)^2}{2nl^2} \sqrt{\frac{EI}{\rho A}}
\]

Higher Order Free-Free Beam Modes

<table>
<thead>
<tr>
<th>Mode</th>
<th>$n$</th>
<th>Nodal Points</th>
<th>$k_nl$</th>
<th>$f_n/l_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fundamental</td>
<td>1</td>
<td>2</td>
<td>4.730</td>
<td>1.000</td>
</tr>
<tr>
<td>1st Harmonic</td>
<td>2</td>
<td>3</td>
<td>7.853</td>
<td>2.757</td>
</tr>
<tr>
<td>2nd Harmonic</td>
<td>3</td>
<td>4</td>
<td>10.996</td>
<td>5.404</td>
</tr>
<tr>
<td>3rd Harmonic</td>
<td>4</td>
<td>5</td>
<td>14.134</td>
<td>8.932</td>
</tr>
<tr>
<td>4th Harmonic</td>
<td>5</td>
<td>6</td>
<td>17.279</td>
<td>13.344</td>
</tr>
</tbody>
</table>

These values of $kl$ correspond to the different modes of vibration.

More than 10x increase
Mode Shape Expression

- The mode shape expression can be obtained by using the fact that \( A=C \) and \( B=D \) into (2), yielding

\[ u_x = \mathcal{A} \left[ \left( \frac{A}{B} \right) (\cosh kx + \cos kx) + (\sinh kx + \sin kx) \right] \]

- Get the amplitude ratio by expanding (3) [the matrix] and solving, which yields

\[ \frac{A}{B} = \frac{\sin k\ell - \sinh k\ell}{\cosh k\ell - \cos k\ell} \]

- Then just substitute the roots for each mode to get the expression for mode shape

Fundamental Mode (n=1)
[Substitute \( k\ell = 4.730 \)]