EE C245 – ME C218
Introduction to MEMS Design
Fall 2019

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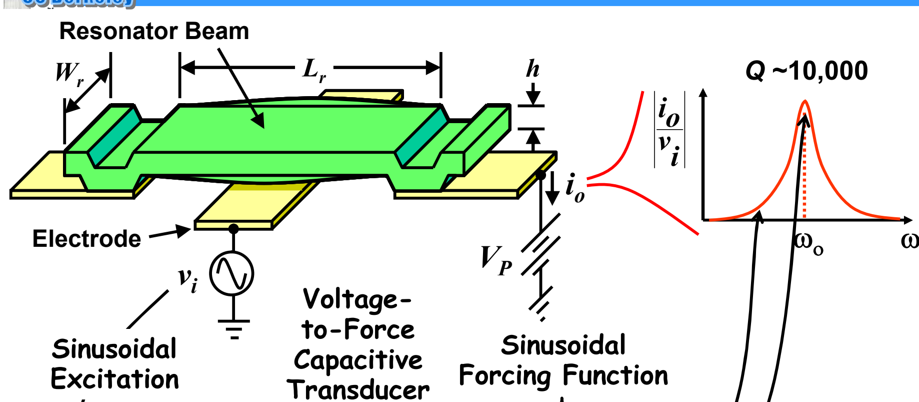
Lecture Module 10: Resonance Frequency

Lecture Outline

- Reading: Senturia, Chpt. 10: §10.5, Chpt. 19
- Lecture Topics:
  - Estimating Resonance Frequency
  - Lumped Mass-Spring Approximation
  - ADXL-50 Resonance Frequency
  - Distributed Mass & Stiffness
  - Folded-Beam Resonator

Estimating Resonance Frequency

Clamped-Clamped Beam μResonator



$$v_i = V_i \cos[\omega_o t] \longrightarrow f_i = \overbrace{F_i \cos[\omega_o t]}$$

- $\omega \neq \omega_o$: small amplitude
- $\omega = \omega_o$: maximum amplitude $\rightarrow$ beam reaches its maximum potential and kinetic energies

Estimating Resonance Frequency

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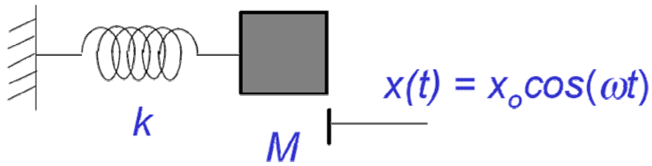
- Assume simple harmonic motion:



- Potential Energy:

$$\mathcal{W}(t) = \frac{1}{2}kx^2(t) = \frac{1}{2}kx_o^{\;2}\cos^2(\omega t)$$

- Kinetic Energy:

$$K(t) = \frac{1}{2}M\dot{x}^2(t) = \frac{1}{2}Mx_o^{\;2}\omega^2\sin^2(\omega t)$$

Estimating Resonance Frequency (cont)

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- Energy must be conserved:
  - Potential Energy + Kinetic Energy = Total Energy
  - Must be true at every point on the mechanical structure

Occurs at peak displacement

Occurs when the beam moves through zero displacement

$$\mathcal{W}_{\max} = \frac{1}{2}kx_o^{\;2} = K_{\max} = \frac{1}{2}M\omega^2 x_o^{\;2}$$

Maximum Potential Energy

Stiffness

Displacement Amplitude

Maximum Kinetic Energy

Mass

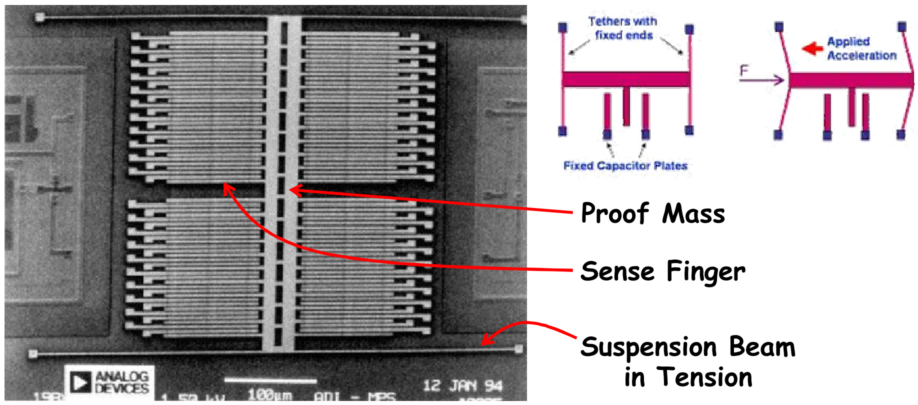
Radian Frequency

- Solving, we obtain for resonance frequency:

$$\boxed{\omega = \sqrt{\frac{k}{M}}}$$

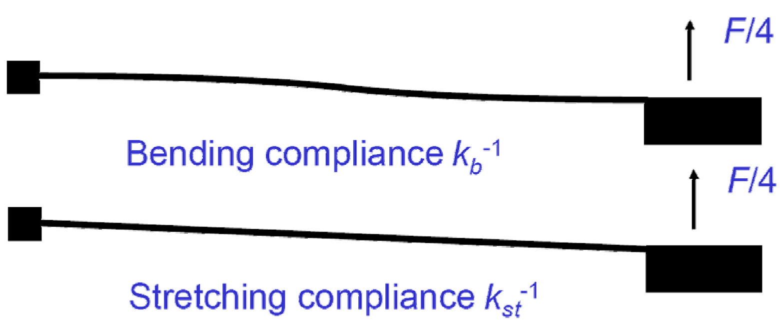
Example: ADXL-50

- The proof mass of the ADXL-50 is many times larger than the effective mass of its suspension beams
  - Can ignore the mass of the suspension beams (which greatly simplifies the analysis)
- Suspension Beam: L = 260 μm, h = 2.3 μm, W = 2 μm



Lumped Spring-Mass Approximation

- Mass is dominated by the proof mass
  - 60% of mass from sense fingers
  - Mass = M = 162 ng (nano-grams)
- Suspension: four tensioned beams
  - Include both bending and stretching terms [A.P. Pisano, BSAC Inertial Sensor Short Courses, 1995-1998]



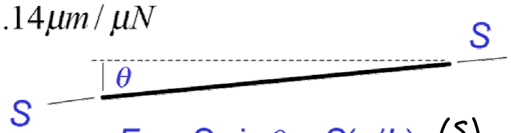
ADXL-50 Suspension Model

- Bending contribution:

$$k_b^{-1} = (1/k_c + 1/k_c) = 2\left[\frac{(L/2)^3}{3E(Wh^3/12)}\right] = \frac{L^3}{EWh^3} = 4.2\mu m/\mu N$$

- Stretching contribution:

$$k_{st}^{-1} = L/S = \frac{L}{\sigma_r Wh} = 1.14\mu m/\mu N$$



$$F_y = S\sin\theta \approx S(x/L) = \underbrace{\left(\frac{S}{L}\right)}_{= k_{st}} x$$

- Total spring constant: *add* bending to stretching (since they are in parallel)

$$k = 4(k_b + k_{st}) = 4(0.24 + 0.88) = 4.5\mu N/\mu m$$

ADXL-50 Resonance Frequency

- Using a lumped mass-spring approximation:

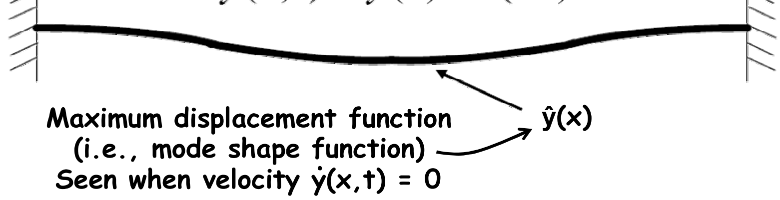
$$f = \frac{1}{2\pi}\sqrt{\frac{k}{M}} = \frac{1}{2\pi}\sqrt{\frac{4.48N/m}{162x10^{-12}kg}} = 26.5kHz$$

- On the ADXL-50 Data Sheet: f_o = 24 kHz
  - Why the 10% difference?
  - Well, it's approximate … plus …
  - Above analysis does not include the frequency-pulling effect of the DC bias voltage across the plate sense fingers and stationary sense fingers … something we'll cover later on …

Distributed Mechanical Structures

- Vibrating structure displacement function:

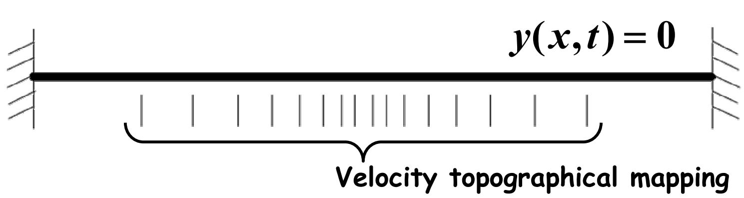
$$y(x,t) = \hat{y}(x)\cos(\omega t)$$



- Procedure for determining resonance frequency:
  - Use the static displacement of the structure as a trial function and find the strain energy W_{max} at the point of maximum displacement (e.g., when t=0, π/ω, …)
  - Determine the maximum kinetic energy when the beam is at zero displacement (e.g., when it experiences its maximum velocity)
  - Equate energies and solve for frequency

Maximum Kinetic Energy

- Displacement: $y(x,t) = \hat{y}(x)\cos[\omega t]$
- Velocity: $v(x,t) = \dfrac{\partial y(x,t)}{\partial t} = -\omega\hat{y}(x)\sin[\omega t]$
- At times t = $\pi/(2\omega)$, $3\pi/(2\omega)$, …



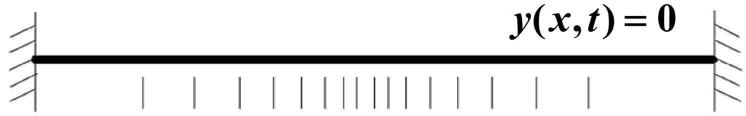
  - The displacement of the structure is y(x,t) = 0
  - The velocity is maximum and all of the energy in the structure is kinetic (since W=0):

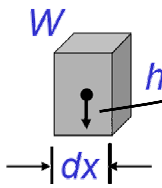
$$v(x,(2n+1)\pi/(2\omega)) = -\omega\hat{y}(x)$$

Maximum Kinetic Energy (cont)

- At times t = π/(2ω), 3π/(2ω), …

$$y(x,t)=0$$





Velocity: $v(x,(2n+1)\pi/(2\omega))=-\omega\hat{y}(x)$

$$dK=\frac{1}{2}\cdot dm\cdot[v(x,t)]^2$$

$$dm=\rho(Wh\cdot dx)$$

- Maximum kinetic energy:

$$K_{\max}=\int_0^L\frac{1}{2}\rho Whdxv^2(x,t')=\int_0^L\frac{1}{2}\rho Wh\omega^2\hat{y}^2(x)dx$$

The Raleigh-Ritz Method

- Equate the maximum potential and maximum kinetic energies:

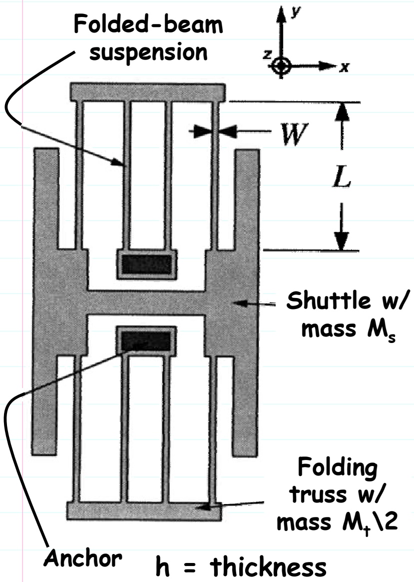
$$K_{\max}=\int_0^L\frac{1}{2}\rho Wh\omega^2\hat{y}^2(x)dx=\mathcal{W}_{\max}$$

- Rearranging yields for resonance frequency:

$$\boxed{\omega=\sqrt{\frac{\mathcal{W}_{\max}}{\int_0^L\frac{1}{2}\rho Wh\hat{y}^2(x)dx}}}$$

ω = resonance frequency
$\mathcal{W}_{max}$ = maximum potential energy
ρ = density of the structural material
W = beam width
h = beam thickness
ŷ(x) = resonance mode shape

Example: Folded-Beam Resonator



- Derive an expression for the resonance frequency of the folded-beam structure at left.

Use Rayleigh-Ritz method.

$$KE_{max} = PE_{max}$$

Kinetic Energy:

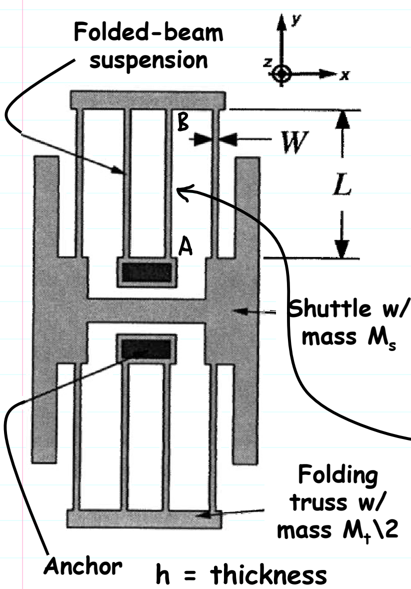
$$KE_{max} = \underbrace{KE_s}_{\text{shuttle}} + \underbrace{KE_t}_{\text{truss}} + \underbrace{KE_b}_{\text{beams}}$$

$$= \frac{1}{2} v_s^2 M_s + \frac{1}{2} v_t^2 M_t + \frac{1}{2} \int v_b^2 dM_b$$

mass of both trusses

Must integrate since the beam velocity is a function of location y!

Get Kinetic Energies



Velocity of the shuttle: $v_s = \omega_o X_o$

Resonance Freq.

Maximum Displacement Amplitude

$$\therefore\ KE_s = \frac{1}{2} v_s^2 M_s = \frac{1}{2}\omega_o^2 X_o^2 M_s$$

Velocity of the truss: $v_t = \frac{1}{2} v_s = \frac{1}{2}\omega_o X_o$

$$\therefore\ KE_t = \frac{1}{2}\left(\frac{1}{2}\omega_o X_o\right)^2 M_t = \frac{1}{8}\omega_o^2 X_o^2 M_t$$

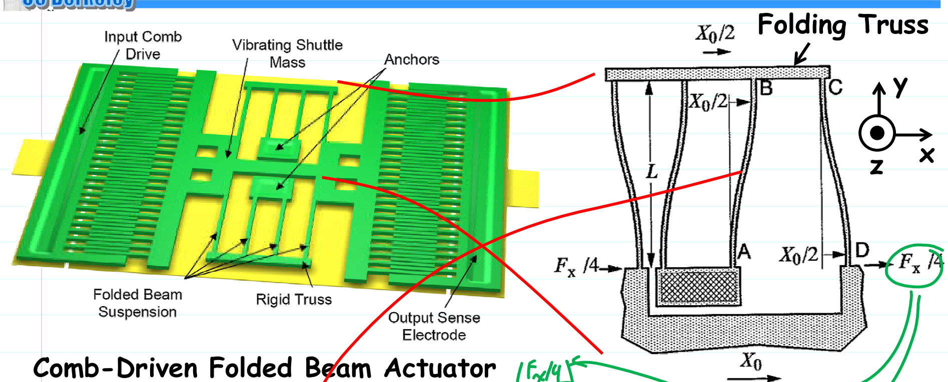
Velocity of the beam segments:

⇒ assume the mode shape is the same as the static displacement shape

⇒ For segment AB:

$$\hat{x}(y) = \frac{F_x}{48 E I_z}\left(3Ly^2 - 2y^3\right) \quad 0 \le y \le L \quad (1)$$

Folded-Beam Suspension



Comb-Driven Folded Beam Actuator

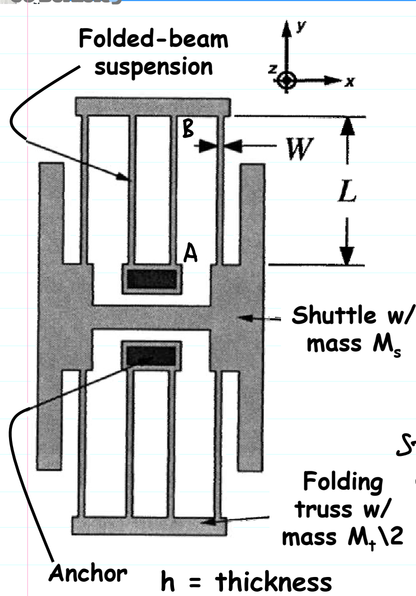
$$\frac{F_x/4}{12EI_z}$$

$$\hat{x}(y) = \frac{F_x}{48EI_z}\left(3Ly^2 - 2y^3\right) \qquad 0 \le y \le L$$

Case: $y=0$ $\quad \hat{x}(y=0) = 0$ ✓

Case: $y=L_c$ $\quad \hat{x}(y=L_c) = \frac{F_x}{48EI_z}L^3 \rightarrow k = \frac{(F_x/4)}{x} = \frac{12EI_z}{L^3} = \frac{k_c}{2}$ ✓

Get Kinetic Energies (cont)



At $y=L$: $x(L) = \frac{X_0}{2} = \frac{F_x L^3}{48EI_z}$

Substituting into (1):

$$\hat{x}(y) = \frac{X_0}{2}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]$$

which yields for velocity:

$$v_b(y)\Big|_{[AB]} = \frac{X_0}{2}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]\omega_0$$

Plugging into the expression for KE_b:

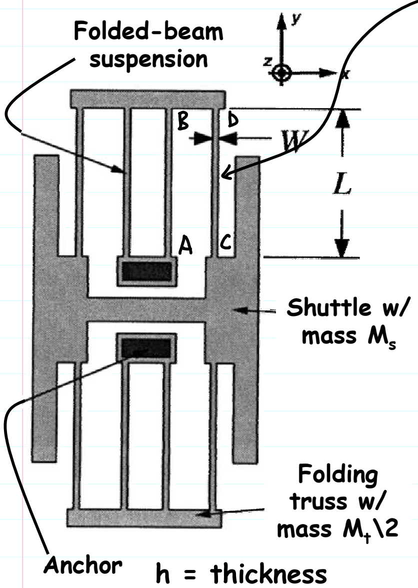
$$KE_{[AB]} = \frac{1}{2}\int_0^L \frac{X_0^2\omega_0^2}{4}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]^2 dM_{[AB]}$$

Static mass of beam [AB]

$$= \frac{X_0^2\omega_0^2 M_{[AB]}}{8L}\int_0^L \left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]^2 dy$$

$$KE_{[AB]} = \frac{13}{280}X_0^2\omega_0^2 M_{[AB]}$$

Get Kinetic Energies (cont)



For segment CD:

$$v_b(y)\big|_{[CD]} = X_0\left[1 - \frac{3}{2}\left(\frac{y}{L}\right)^2 + \left(\frac{y}{L}\right)^3\right]\omega_0$$

Thus:

$$KE_{[CD]} = \frac{X_0^2\omega_0^2 M_{[CD]}}{2L}\int_0^L \left[1 - \frac{3}{2}\left(\frac{y}{L}\right)^2 + \left(\frac{y}{L}\right)^3\right] dy$$

$$KE_{[CD]} = \frac{83}{280} X_0^2\omega_0^2 M_{[CD]}$$

Static mass of beam [CD]

Let $M_b \triangleq$ total mass of the 8 beams.

Then: $M_{[AB]} = M_{[CD]} = \frac{1}{8}M_b$

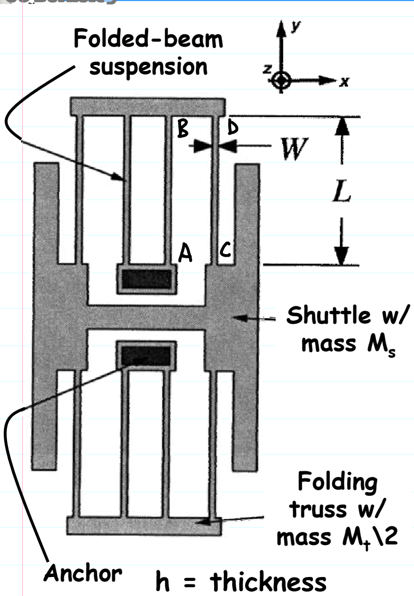
Thus:

$$KE_b = 4KE_{[AB]} + 4K_{[CD]} = \frac{6}{35}X_0^2\omega_0^2 M_b$$

and

$$KE_{max} = X_0^2\omega_0^2\left[\frac{1}{2}M_s + \frac{1}{8}M_t + \frac{6}{35}M_b\right]$$

Get Potential Energy & Frequency



PE_{max} is simply the work done to achieve maximum deflection:

$$PE_{max} = \frac{1}{2}\overset{=k_c}{k_x} X_0^2$$

Thus, using Raleigh-Ritz:

$$KE_{max} = PE_{max}$$

$$\cancel{X_0^2}\omega_0^2\left[\frac{1}{2}M_s + \frac{1}{8}M_t + \frac{6}{35}M_b\right] = \frac{1}{2}\overset{=k_c}{k_x}\cancel{X_0^2}$$

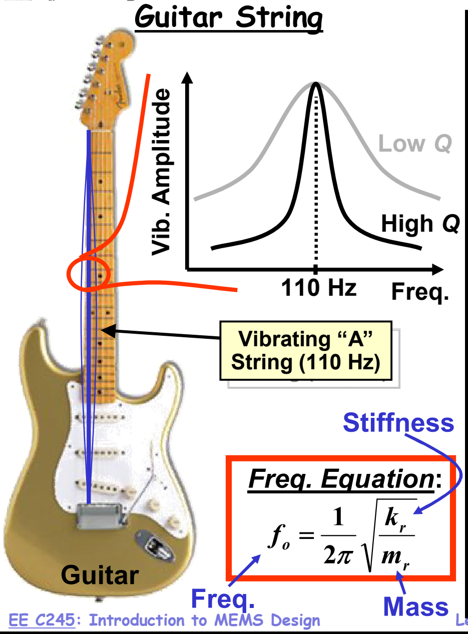
$$\boxed{\omega_0 = \left[\frac{\overset{=k_c}{k_x}}{M_{eq}}\right]^{1/2} \quad \text{where } M_{eq} = M_s + \frac{1}{4}M_t + \frac{12}{35}M_b}$$

(Resonance Frequency of a Folded-Beam Suspended Shuttle)

Brute Force Methods for Resonance Frequency Determination

Basic Concept: Scaling Guitar Strings

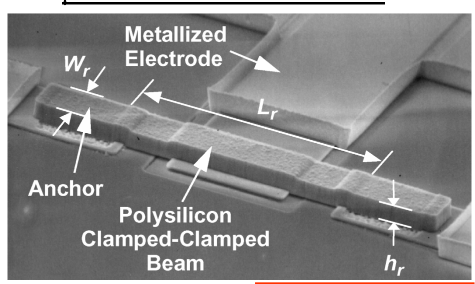
Guitar String



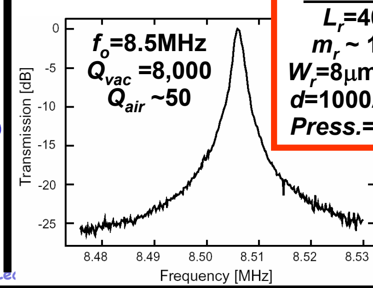
Freq. Equation:

$$f_o = \frac{1}{2\pi}\sqrt{\frac{k_r}{m_r}}$$

μMechanical Resonator

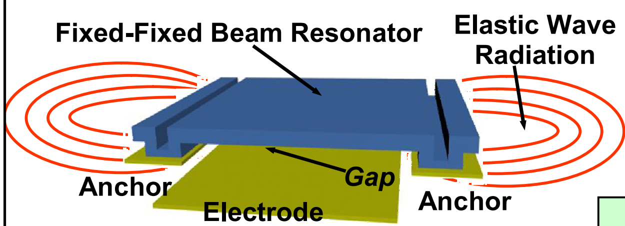


[Bannon 1996]



Performance:
L_r=40.8μm
m_r ~ 10^{-13} kg
W_r=8μm, h_r=2μm
d=1000Å, V_P=5V
Press.=70mTorr

Anchor Losses

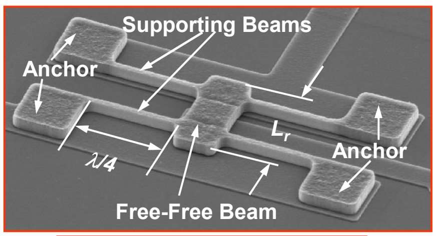


Q = 300 at 70MHz

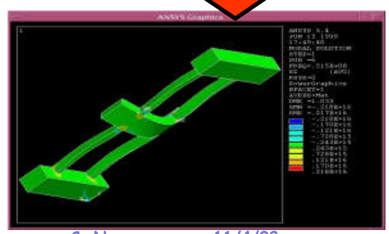
Problem: direct anchoring to the substrate ⇨ anchor radiation into the substrate ⇨ lower Q

Solution: support at motionless nodal points ⇨ isolate resonator from anchors ⇨ less energy loss ⇨ higher Q

Free-Free Beam Resonator

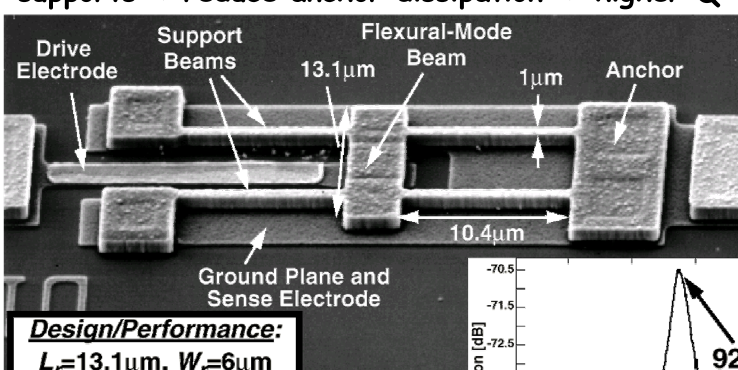


Q = 15,000 at 92MHz

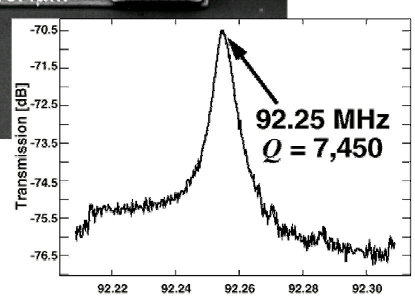


92 MHz Free-Free Beam μResonator

- Free-free beam μmechanical resonator with non-intrusive supports ⇨ reduce anchor dissipation ⇨ higher Q

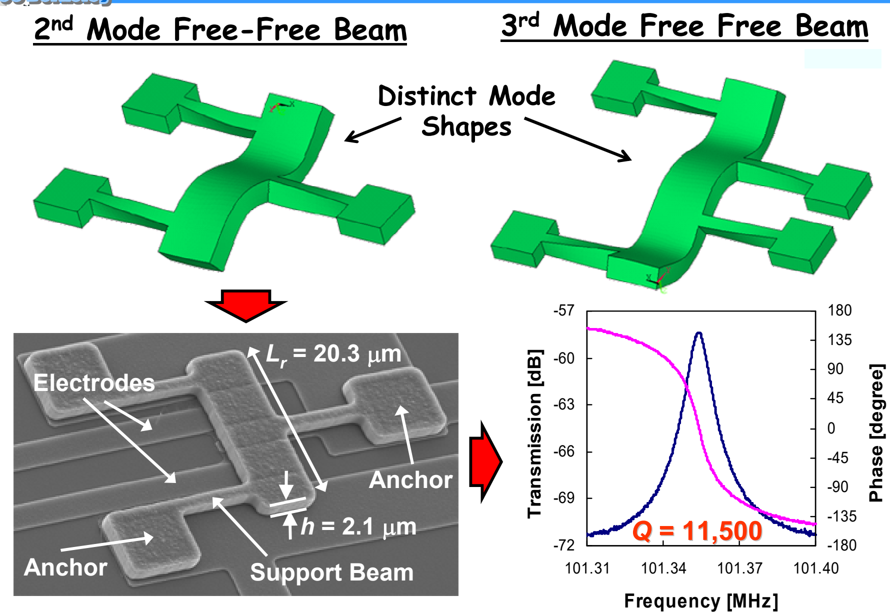


Design/Performance:
L_r=13.1μm, W_r=6μm
h=2μm, d=1000Å
V_P=28-76V, W_e=2.8μm
f_o~92.25MHz
Q~7,450 @ 10mTorr

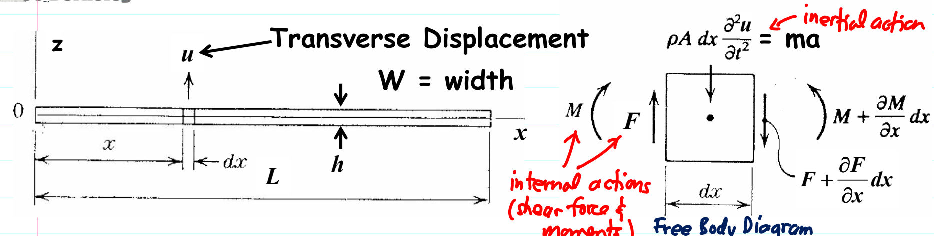


[Wang, Yu, Nguyen 1998]

Higher Order Modes for Higher Freq.



Flexural-Mode Beam Wave Equation



$\rho A\, dx \frac{\partial^2 u}{\partial t^2}$ = ma ← inertial action

- Derive the wave equation for transverse vibration:

Dynamic Equilibrium Condition for Forces in the y-direction: $\cancel{M} + F\frac{dx}{2} + F\frac{dx}{2} + \cancel{\frac{dF}{dx}\frac{dx}{2}} - \cancel{M} - \frac{\partial M}{\partial x}dx = 0$ (neg.)

$$F - \left(F + \frac{\partial F}{\partial x}dx\right) - \rho A dx \frac{\partial^2 u}{\partial t^2} = 0 \quad (1)$$ ← neglect the $\frac{\partial F}{\partial x} dx$ term

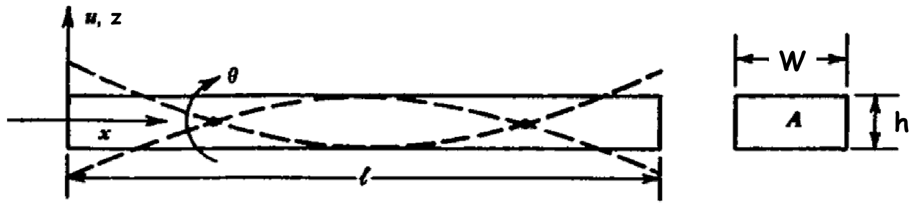
and the moment equilibrium condition: $-F dx + \frac{\partial M}{\partial x} dx \approx 0$ (2)

Combining (1) & (2):

$$\frac{\partial^2 M}{\partial x^2} dx = -\rho A dx \frac{\partial^2 u}{\partial t^2} \longrightarrow \frac{\partial^2}{\partial x^2}\left(-EI\frac{\partial^2 u}{\partial x^2}\right) = -\rho A \frac{\partial^2 u}{\partial t^2} \longrightarrow \boxed{\frac{\partial^2 u}{\partial t^2} = \left(\frac{EI}{\rho A}\right)\frac{\partial^4 u}{\partial x^4} \quad I_y = \frac{Wh^3}{12}}$$

$$\left[\frac{\partial^2 u}{\partial x^2} = -\frac{M}{EI} \rightarrow M = -EI\frac{\partial^2 u}{\partial x^2}\right]$$

Example: Free-Free Beam



- Determine the resonance frequency of the beam
- Specify the lumped parameter mechanical equivalent circuit
- Transform to a lumped parameter electrical equivalent circuit
- Start with the flexural-mode beam equation:

$$\frac{\partial^2 u}{\partial t^2} = \left(\frac{EI}{\rho A}\right)\frac{\partial^4 u}{\partial x^4}$$

Free-Free Beam Frequency

- Substitute $u = u_1 e^{j\omega t}$ into the wave equation:

$$\frac{\partial^4 u}{\partial x^4} = \left(\omega^2 \frac{\rho A}{EI}\right) u \qquad (1)$$

- This is a 4th order differential equation with solution:

$$u(x) = \mathscr{A}\cosh kx + \mathscr{B}\sinh kx + \mathscr{C}\cos kx + \mathscr{D}\sin kx \qquad (2)$$

Gives the mode shape during resonance vibration.

- Boundary Conditions:

At $x = 0$	At $x = \ell$	
$\frac{\partial^2 u}{\partial x^2} = 0$	$\frac{\partial^2 u}{\partial x^2} = 0$	$M = 0$ (Bending moment)
$\frac{\partial^3 u}{\partial x^3} = 0$	$\frac{\partial^3 u}{\partial x^3} = 0$	$\frac{\partial M}{\partial x} = 0$ (Shearing force)

Free-Free Beam Frequency (cont)

- Applying B.C.'s, get A=C and B=D, and

$$\begin{bmatrix} (\cosh k\ell - \cos k\ell) & (\sinh k\ell - \sin k\ell) \\ (\sinh k\ell + \sin k\ell) & (\cosh k\ell - \cos k\ell) \end{bmatrix} \begin{bmatrix} \mathscr{A} \\ \mathscr{B} \end{bmatrix} = 0 \quad (3)$$

- Setting the determinant = 0 yields

$$\cos k\ell = \frac{1}{\cosh k\ell}$$

- Which has roots at

$$k_1\ell = 4.730 \qquad k_2\ell = 7.853 \qquad k_3\ell = 10.996$$

These values of $k_n\ell$ correspond to the different modes of vibration!

- Substituting (2) into (1) finally yields:

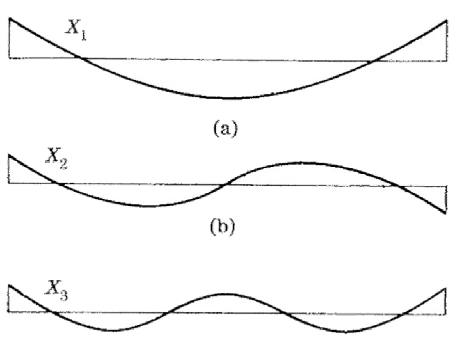
$$k^4 = \frac{\rho A}{EI}\omega^2 \longrightarrow \boxed{f_n = \frac{(k_n\ell)^2}{2\pi\ell^2}\sqrt{\frac{EI}{\rho A}}}$$

Free-Free Beam Frequency Equation

Higher Order Free-Free Beam Modes

Mode	n	Nodal Points	$k_n\ell$	f_n/f_1
Fundamental (f_1)	1	2	4.730	1.000
1st Harmonic	2	3	7.853	2.757
2nd Harmonic	3	4	10.996	5.404
3rd Harmonic	4	5	14.137	8.932
4th Harmonic	5	6	17.279	13.344

← More than 10x increase



(a) Fundamental Mode (n=1)

(b) 1st Harmonic (n=2)

2nd Harmonic (n=3)

Mode Shape Expression

- The mode shape expression can be obtained by using the fact that A=C and B=D into (2), yielding

$$u_x = \mathscr{B}\left[\left(\frac{\mathscr{A}}{\mathscr{B}}\right)(\cosh kx + \cos kx) + (\sinh kx + \sin kx)\right]$$

- Get the amplitude ratio by expanding (3) [the matrix] and solving, which yields

$$\frac{\mathscr{A}}{\mathscr{B}} = \frac{\sin k\ell - \sinh k\ell}{\cosh k\ell - \cos k\ell}$$

- Then just substitute the roots for each mode to get the expression for mode shape

Fundamental Mode (n=1)
[Substitute $k_1\ell = 4.730$]

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