Lecture Outline

* Reading: Senturia, Chpt. 5
* Lecture Topics:
  - Lumped Mass
  - Lumped Stiffness
  - Lumped Damping
  - Lumped Mechanical Equivalent Circuits
  - Electromechanical Analogies
Lumped Parameter Mechanical Equivalent Circuit

Equivalent Dynamic Mass

- Once the mode shape is known, the lumped parameter equivalent circuit can then be specified.
- Determine the equivalent mass at a specific location $x$ using knowledge of kinetic energy and velocity.

$$\text{Maximum Kinetic Energy} = \int_0^x V^2(x) \, dx$$

$$\text{Equivalent Mass} = M_{eq} = \frac{K.E.}{\frac{1}{2} V_x^2} = \frac{1}{2} \rho A \frac{\int_0^x V^2(x) \, dx}{\frac{1}{2} V_x^2}$$

Maximum Velocity @ location $x$
Equivalent Dynamic Mass

- For the folded-beam structure, we've already determined the maximum kinetic energy.
- And in our resonance frequency analysis, we've already determined expressions for velocity.

Equivalent Dynamic Stiffness & Damping

- Stiffness then follows directly from knowledge of mass and resonance frequency.
  \[ \omega_0 = \sqrt{\frac{K_{eq}(x)}{M_{eq}(x)}} \quad \Rightarrow \quad K_{eq}(x) = \omega_0^2 M_{eq}(x) \]

- And damping also follows readily from knowledge of Q or other loss measurands.
  \[ Q = \frac{\omega_0 M_{eq}(x)}{C_{eq}(x)} \quad \Rightarrow \quad C_{eq}(x) = \frac{\omega_0 M_{eq}(x)}{Q} \]

- With mass, stiffness, and damping \(\Rightarrow\) lumped parameter equivalent circuit.
Get Potential Energy & Frequency

Folded-beam suspension

\[ W = 2 \mu m \]

\[ L = 100 \mu m \]

Shuttle w/ mass \( M_s \)

Folding truss w/ mass \( M_t \)

Anchor

\[ h = \text{thickness} = 2 \mu m \]

\[ K_{eq(shuttle)} = 19.2 \text{ N/m} \]

\[ M_{eq(shuttle)} = 8.64 \times 10^{-11} \text{ kg} \]

\[ C_{eq(shuttle)} = 4.08 \times 10^{-10} \text{ kg/s} \]

\[ K_{eq(truss)} = 4.8 \text{ N/m} \]

\[ M_{eq(truss)} = 2.16 \times 10^{-11} \text{ kg} \]

\[ C_{eq(truss)} = 1.02 \times 10^{-10} \text{ kg/s} \]

Electromechanical Analogies

Current \[ i = q \]

Voltage \[ V(t) \]

\[ F(t) \]

\[ X(t) \]

\[ r_x \]

\[ c_x \]

\[ m_w \]

\[ k \]

\[ c \]

\[ T \]

\[ \omega \]

Equation of Motion:

\[ m_{eq} \ddot{x} + c_{eq} \dot{x} + k_{eq} x = F(t) \]

\[ \Rightarrow \text{using phaser concepts}: \]

\[ F = j \omega m_{eq} \dot{x} + k_{eq} \dot{x} + c_{eq} x \]

\[ \Rightarrow \text{by analogy}: \]

\[ F \rightarrow N \]

\[ m_{eq} \rightarrow L \]

\[ c_{eq} \rightarrow r_c \]

\[ X \rightarrow i \]

\[ k_{eq} \rightarrow \frac{1}{C_x} \]
Electromechanical Analogies (cont)

- Mechanical-to-electrical correspondence in the current analogy:

<table>
<thead>
<tr>
<th>Mechanical Variable</th>
<th>Electrical Variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>Damping, $c$</td>
<td>Resistance, $R$</td>
</tr>
<tr>
<td>Stiffness$^{-1}$, $k^{-1}$</td>
<td>Capacitance, $C$</td>
</tr>
<tr>
<td>Mass, $m$</td>
<td>Inductance, $L$</td>
</tr>
<tr>
<td>Force, $f$</td>
<td>Voltage, $V$</td>
</tr>
<tr>
<td>Velocity, $v$</td>
<td>Current, $I$</td>
</tr>
</tbody>
</table>

Bandpass Biquad Transfer Function

\[
F = j\omega m_0 x + \frac{k_{eq}}{j\omega} x + C_{eq} \dot{x}
\]

\[
F = (j\omega) (j\omega) m_{eq} + \frac{k_{eq}}{j\omega} (j\omega) + C_{eq} (j\omega)
\]

\[
\frac{X}{F} (j\omega) = \frac{1}{j\omega m_{eq}} \left[ -\omega^2 \frac{m_{eq}}{k_{eq}} + 1 + \frac{C_{eq}\omega}{k_{eq}} \right]^{-1} = \frac{1}{j\omega m_{eq}} \left[ (\omega_0^2)^2 + 1 + \frac{\omega^2}{\omega_0^2} \right]^{-1}
\]

\[
\left[ \frac{k_{eq}}{m_{eq}}, \frac{Q}{\omega_0}, \frac{Q_{w_0}}{C_{eq}}, \frac{C_{eq}}{k_{eq}} \right]
\]
3CC $3\lambda/4$ Bridged μMechanical Filter

**Performance:**
- $f_o=9$ MHz, $BW=20$ kHz, $PBW=0.2\%$
- $I.L.=2.79$ dB, Stop. Rej.=51 dB
- $20$ dB S.F.=1.95, $40$ dB S.F.=6.45

**Design:**
- $L_r=40\mu m$
- $W_r=6.5\mu m$
- $h_r=2\mu m$
- $C_r=3.5\mu m$
- $L_b=1.6\mu m$
- $V_p=10.47V$
- $P=-5$ dBm
- $R_Q=R_{oo}=12k\Omega$

**Micromechanical Filter Circuit**

- Bridging Beam
- Coupling Beam
- Resonator

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Micromechanical Filter Circuit

Input

Bridging Beam

Coupling Beam

Resonator

Output

\[ \frac{V_o}{V_i} \]

\[ \frac{V_o}{V_i} \]

\[ \frac{V_o}{V_i} \]

\[ \frac{V_o}{V_i} \]
Micromechanical Filter Circuit

3CC 3\lambda/4 Bridged \mu\text{Mechanical Filter}

Performance:
\( f_0 = 9\text{MHz} \), \( BW = 20\text{kHz} \), \( PBW = 0.2\% \)
\( I.L. = 2.79\text{dB} \), \( \text{Stop. Rej.} = 51\text{dB} \)
20dB S.F. = 1.95, 40dB S.F. = 6.45

Design:
\( L_r = 40\mu\text{m} \)
\( W_f = 6.5\mu\text{m} \)
\( h = 2\mu\text{m} \)
\( L_c = 3.5\mu\text{m} \)
\( L_b = 1.6\mu\text{m} \)
\( V_p = 10.47\text{V} \)
\( P = -5\text{dBm} \)
\( R_{Q_o} = R_{Q_i} = 12\text{k}\Omega \)
Beam Resonator Equivalent Circuits
(Pretty Much the Same Stuff)

Equivalent Dynamic Mass

* Once the mode shape is known, the lumped parameter equivalent circuit can then be specified
* Determine the equivalent mass at a specific location $x$ using knowledge of kinetic energy and velocity

$$M_{eq, x} = \frac{K.E.}{\frac{1}{2} V_x^2} = \frac{1}{2} \rho A \int_0^x V^2(x) \, dx$$

Maximum Kinetic Energy

Maximum Velocity @ location $x$
**Equivalent Dynamic Mass**

* We know the mode shape, so we can write expressions for displacement and velocity at resonance:

\[
\text{Displacement: } u(x) = B \left[ 9 (\cosh kx + \cos k\pi x) + (\sinh kx + \sin k\pi x) \right], \quad s^2 \frac{A}{B}
\]

\[
\left[ V(x) = \omega_0^2 u(x) \right] \Rightarrow M_{eq}(x) = \frac{k_{F_{max}}}{\frac{1}{2} \omega_0^2} \mathcal{L}_{x} \frac{w^2}{w(x)} \frac{dx}{x^2} = \frac{1}{2} \rho A \mathcal{L}_{x} \frac{w^2}{w(x)} \frac{dx}{x^2}
\]

\[
M_{eq}(x) = \frac{\rho A}{B} \int_0^L \frac{1}{2} \left[ (\cosh kx + \cos k\pi x) + (\sinh kx + \sin k\pi x) \right]^2 dx
\]

**Equivalent Dynamic Stiffness & Damping**

* Stiffness then follows directly from knowledge of mass and resonance frequency:

\[
\omega_0: \frac{K_{eq}(x)}{\sqrt{M_{eq}(x)}} \rightarrow K_{eq}(x) = \omega_0^2 M_{eq}(x)
\]

* And damping also follows readily:

\[
Q: \frac{\omega_0 M_{eq}(x)}{C_{eq}(x)} \rightarrow C_{eq}(x) = \frac{\omega_0 M_{eq}(x)}{Q} = \frac{1}{k_{eq}(x) M_{eq}(x)}
\]
Equivalent Lumped Mechanical Circuit

\[ K_{eq}(x) = \omega_0^2 M_{eq}(x) \]

\[ M_{eq}(x) = \frac{\rho A}{\omega} \int_{0}^{l} [u(x')]^2 dx' \]

\[ C_{eq}(x) = \frac{\omega_0 M_{eq}(x)}{Q} \]

Example: Polysilicon w/ \( \ell = 14.9 \mu m, W = 6 \mu m, h = 2 \mu m \to 70 \text{ MHz} \)

\[ K_{eq}(0) = 19,927 \text{ N/m} \]
\[ M_{eq}(0) = 1.03 \times 10^{-13} \text{ kg} \]
\[ C_{eq}(0) = 5.66 \times 10^{-9} \text{ kg/s} \]

\[ K_{eq}(l/2) = 53,938 \text{ N/m} \]
\[ M_{eq}(l/2) = 2.78 \times 10^{-13} \text{ kg} \]
\[ C_{eq}(l/2) = 1.53 \times 10^{-8} \text{ kg/s} \]