Lecture Outline

- Reading: Senturia, Chpt. 14, Chpt. 16, Chpt. 21
- Lecture Topics:
  - Gyroscopes
  - Gyro Circuit Modeling
  - Minimum Detectable Signal (MDS)
    - Noise
    - Angle Random Walk (ARW)
Gyroscopes

**Classic Spinning Gyroscope**

- A gyroscope measures rotation rate, which then gives orientation → very important, of course, for navigation
- Principle of operation based on conservation of momentum
- Example: classic spinning gyroscope

Rotor will preserve its angular momentum (i.e., will maintain its axis of spin) despite rotation of its gimbed chassis
Vibratory Gyroscopes

- Generate momentum by vibrating structures
- Again, conservation of momentum leads to mechanisms for measuring rotation rate and orientation
- Example: vibrating mass in a rotating frame

\[
\begin{align*}
\text{Mass at rest} & \quad \text{Driven into vibration along the } y\text{-axis} \\
\text{y-displaced mass} & \quad \text{Get an } x' \text{ component of motion} \\
\text{Capacitance between mass and frame} & = \text{constant}
\end{align*}
\]

Principle of Operation

- Tuning Fork Gyroscope:
  - Input Rotation
  - Driven Vibration @ \( f_c \)
  - Coriolis (Sense) Response
  - Coriolis Torque
  - Detect motion out-of-the plane of the tuning fork as rotation!
**Basic Vibratory Gyroscope Operation**

**Principle of Operation**
- Tuning Fork Gyroscope:

**Drive/Sense Response Spectra:**

\[
\omega \quad \text{Amplitude} \quad f_0 (@ T_1)
\]

- Coriolis Acceleration
- Driven Velocity
- Rotation Rate
- Beam Mass
- Sense Frequency

\[
\tilde{a}_c = 2\tilde{v} \times \Omega
\]

**Vibratory Gyroscope Performance**

**Principle of Operation**
- Tuning Fork Gyroscope:

\[
\tilde{x} = \frac{\tilde{F}_c}{k} = \frac{m\tilde{a}_c}{k} = \frac{\tilde{a}_c}{\omega_c^2}
\]

- Beam Mass
- Beam Stiffness
- Sense Frequency
- Driven Velocity

\[
\tilde{a}_c = 2\tilde{v} \times \Omega
\]

*To maximize the output signal \( x \), need:
- Large sense-axis mass
- Small sense-axis stiffness
- (Above together mean low resonance frequency)
- Large drive amplitude for large driven velocity (so use comb-drive)
- If can match drive freq. to sense freq., then can amplify output by \( Q \) times
MEMS-Based Gyroscopes

**MEMS-Based Tuning Fork Gyroscope**

- In-plane drive and sense modes pick up z-axis rotations
- Mode-matching for maximum output sensitivity
- From [Zaman, Ayazi, et al, MEMS'06]
MEMS-Based Tuning Fork Gyroscope

• Drive and sense axes must be stable or at least track one another to avoid output drift

Need: small or matched drive and sense axis temperature coefficients to suppress drift

Problem: if drive frequency changes relative to sense frequency, output changes $\Rightarrow$ bias drift

[Zaman, Ayazi, et al, MEMS'06]
Mode Matching for Higher Resolution

- For higher resolution, can try to match drive and sense axis resonance frequencies and benefit from Q amplification.

**Problem:** mismatch between drive and sense frequencies → even larger drift!

**Need:** small or matched drive and sense axis temperature coefficients to make this work.

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Issue: Zero Rate Bias Error

- Imbalances in the system can lead to zero rate bias error.

**Drive imbalance** ⇒ off-axis motion of the proof mass

**Mass imbalance** ⇒ off-axis motion of the proof mass

**Quadrature output signal** that can be confused with the Coriolis acceleration

**Output signal** in phase with the Coriolis acceleration
### Nuclear Magnetic Resonance Gyroscope

* The ultimate in miniaturized spinning gyroscopes?  
  From CSAC, we may now have the technology to do this

- Better if this is a noble gas nucleus (rather than e-), since nuclei are heavier ⇔ less susceptible to B field  
  Soln: Spin polarize Xe$^{129}$ nuclei by first polarizing e- of Rb$^{87}$ (a la CSAC), then allowing spin exchange

- Challenge: suppressing the effects of B field

![Diagram of nuclear magnetic resonance gyroscope](image)

### MEMS-Based Tuning Fork Gyroscope

- (-) Sense Output Current  
  (+) Sense Output Current

![Diagram of MEMS-based tuning fork gyroscope](image)

[Zaman, Ayazi, et al, MEMS'06]
Determining Sensor Resolution

MEMS-Based Tuning Fork Gyroscope

[Zaman, Ayazi, et al, MEMS'06]
Drive Axis Equivalent Circuit

- Generates drive displacement velocity $x_d$ to which the Coriolis force is proportional
- To Sense Amplifier (for synchronization)

**Drive-to-Sense Transfer Function**

Rotation-Induced Coriolis Force:

$$\mathbf{a}_s = 2\omega \times \mathbf{x}_d \times \mathbf{\Omega}$$

Acts in the sense mode direction

$$a_s = 2\omega \chi_d \Omega \sin 90^\circ$$
**Gyro Readout Equivalent Circuit**
(for a single tine)

\[ F_c = m\ddot{a}_c = m \cdot (2\ddot{x}_d \times \Omega) \]

- Gyro Sense Element
- Output Circuit

\[ \eta_e \cdot 1 \]

\[ V_0 \]

**Noise Sources**

\[ \begin{align*}
    I_x & = c_x \int \frac{r_x}{f_r} \, dt \\
    R_f & = \int \frac{r_x}{f_r} \, dt
\end{align*} \]

**Easiest to analyze if all noise sources are summed at a common node**

**Minimum Detectable Signal (MDS)**

- Minimum Detectable Signal (MDS): Input signal level when the signal-to-noise ratio (SNR) is equal to unity

\[ \text{Sensor} \quad \text{Scale Factor} \quad \text{Circuit} \quad \text{Gain} \quad \text{Output} \]

- The sensor scale factor is governed by the sensor type
- The effect of noise is best determined via analysis of the equivalent circuit for the system
**Move Noise Sources to a Common Point**

- Move noise sources so that all sum at the input to the amplifier circuit (i.e., at the output of the sense element)
- Then, can compare the output of the sensed signal directly to the noise at this node to get the MDS

**Gyro Readout Equivalent Circuit**

(for a single tine)

- Easiest to analyze if all noise sources are summed at a common node
Gyro Readout Equivalent Circuit
(for a single tine)

\[ F_c = m\ddot{a}_c = m \cdot (2\dddot{x}_d \times \Omega) \]

Noise Sources

\[ i_x \]
\[ c_x \]
\[ f_r \]
\[ r_x \]
\[ \eta \]
\[ i_{eq} \]
\[ v_{eq} \]

Gyro Sense Element
Output Circuit

Signal Conditioning Circuit
(Transresistance Amplifier)

\[ V_0 \]

\[ i_{eq}^2 \]

\[ v_{eq}^2 \]

* Here, \( v_{eq}^2 \) and \( i_{eq}^2 \) are equivalent input-referred voltage and current noise sources
Noise

- Noise: Random fluctuation of a given parameter $I(t)$
- In addition, a noise waveform has a zero average value

- We can't handle noise at instantaneous times
- But we can handle some of the averaged effects of random fluctuations by giving noise a power spectral density representation
- Thus, represent noise by its mean-square value:

Let $i(t) = I(t) - I_D$

Then $\overline{i^2} = \left(\overline{I - I_D}\right)^2 = \lim_{T \to \infty} \frac{1}{T} \int_0^T |I - I_D|^2 \, dt$

Noise Spectral Density

- We can plot the spectral density of this mean-square value:

\[ \overline{i^2} \text{ [units}^2/\text{Hz}] \]

One-sided spectral density
→ used in circuits
→ measured by spectrum analyzers

Two-sided spectral density
(1/2 the one-sided)
Often used in systems courses

\[ \overline{i^2} = \text{integrated mean-square noise spectral density over all frequencies (area under the curve)} \]
Circuit Noise Calculations

**Inputs**

- Deterministic
  - \( v_i(j \omega) \)
  - \( S_i(\omega) \)

**Outputs**

- Deterministic
  - \( v_o(j \omega) = H(j \omega) v_i(j \omega) \)

- Random
  - \( S_o(\omega) = |H(j \omega)H^*(j \omega)|S_i(\omega) = |H(j \omega)|^2S_i(\omega) \)

\[
\sqrt{S_o(\omega)} = |H(j \omega)|\sqrt{S_i(\omega)}
\]

* Deterministic: \( v_o(j \omega) = H(j \omega) v_i(j \omega) \)

* Random: \( S_o(\omega) = H(j \omega)H^*(j \omega)S_i(\omega) = |H(j \omega)|^2S_i(\omega) \)

Handling Noise Deterministically

* Can do this for noise in a tiny bandwidth (e.g., 1 Hz)

\[
\frac{v_{n1}^2}{\Delta f} = S_1(f) \quad v_{n1} = \sqrt{S_1(f) \cdot B}
\]

Can approximate this by a sinusoidal voltage generator (especially for small B, say 1 Hz)

\[
S_n(j \omega)
\]

**Why?** Neither the amplitude nor the phase of a signal can change appreciably within a time period \( 1/B \).

[This is actually the principle by which oscillators work → oscillators are just noise going through a tiny bandwidth filter]
Systematic Noise Calculation Procedure

1. For \( \overline{i_{n1}^2} \), replace it with a deterministic source of value \( \overline{i_{n1}^2} \).

2. Calculate \( v_{on1}(\omega) = \overline{i_{n1}(\omega)H(j\omega)} \) (treating it like a deterministic signal).

3. Determine \( \overline{v_{on1}^2} = \overline{i_{n1}^2} \cdot |H(j\omega)|^2 \).

4. Repeat for each noise source: \( \overline{v_{on2}^2}, \overline{v_{on3}^2}, \overline{v_{on4}^2} \).

5. Add noise power (mean square values)

\[
\overline{v_{onTOT}^2} = \overline{v_{on1}^2} + \overline{v_{on2}^2} + \overline{v_{on3}^2} + \overline{v_{on4}^2} + \cdots
\]

\[
\overline{v_{onTOT}} = \sqrt{\overline{v_{on1}^2} + \overline{v_{on2}^2} + \overline{v_{on3}^2} + \overline{v_{on4}^2} + \cdots}
\]

Total rms value
Determining Sensor Resolution

Example: Gyro MDS Calculation

\[ \vec{F}_c = m \bar{\vec{a}}_c = m \cdot (2 \vec{x}_d \times \Omega) \]

* The gyro sense presents a large effective source impedance
  \( \Rightarrow \) Currents are the important variable; voltages are "opened" out
  \( \Rightarrow \) Must compare \( i_o \) with the total current noise \( i_{eqTOT} \) going into the amplifier circuit

\[ R_f \]

\[ V_0 \]
Example: Gyro MDS Calculation (cont)

First, find the rotation to $i_o$ transfer function:

$$\dot{x}_s = \frac{\omega_0 Q}{k_x} \Theta_x(j\omega) \frac{F_s}{F_c} = \frac{\omega_0 Q}{k_x} \cdot 2\omega_0 x_0 \sin\omega R\omega \frac{1}{\omega_e}$$

$$\dot{x}_s = 2 \frac{\omega_0}{\omega_e} Q x_0 \Theta(j\omega) R\omega$$

When $\omega \sim \omega_m$, $i_o \sim i_{eq}$:

$$i_o \sim \frac{A_i}{R}$$

$$\downarrow$$

Input-referred noise current entering the sense amplifier into $i_{eq}$

$$i_{eq} = A_i \frac{\omega_0}{2} Q x_0 \eta_e \Theta(j\omega)$$

$$\downarrow$$

Easier to determine directional error as a function of elapsed time.
Example: Gyro MDS Calculation (cont)

\[ F_c = m \ddot{a}_c = m \cdot (2 \dddot{x}_d \times \dddot{\Omega}) \]

\[ I_x e \cdot f_r^2 r_x \eta \cdot 1 \]

\[ i_{eq} \]

\[ v_{eq}^2 \]

\[ R_f \]

\[ V_0 \]

Now, find the \( i_{eqTOT} \) entering the amplifier input:

\[ i_{eqTOT} = i_s + i_{eq} \]

Brownian motion noise of the sense element is determined entirely by the noise in \( r_x \rightarrow \frac{v_{eq}^2}{R_f} \)

Easiest to convert to an all electrical equivalent circuit.

Example: Gyro MDS Calculation (cont)

\[ L_x \quad C_x \quad R_x \]

\[ \frac{i_s}{R_x} = \frac{kT}{R_x} \]

\[ \Theta(j\omega) \]

\[ \frac{\alpha}{\Delta f} = 4K T \cdot \Theta(j\omega) \cdot \Theta(j\omega) \]

Thus:

\[ \frac{\alpha^2}{\Delta f} = \frac{4K T}{R_x} \cdot \Theta(j\omega) \cdot \Theta(j\omega) + \frac{1}{R_x} \]

Learn to get these from EE240.

\[ \alpha \text{ or just get them from a data sheet...} \]
Example ARW Calculation

- **Example Design:**
  - **Sensor Element:**
    - $m = (100\, \mu m)(100\, \mu m)(20\, \mu m)(2300\, kg/m^3) = 4.6 \times 10^{-10}\, kg$
    - $\omega_s = 2\pi(15\, kHz)$
    - $\omega_d = 2\pi(10\, kHz)$
    - $k_s = \omega_s^2m = 4.09\, N/m$
    - $x_d = 20\, \mu m$
    - $Q_s = 50,000$
    - $V_p = 5V$
    - $h = 20\, \mu m$
    - $d = 1\, \mu m$

- **Sensing Circuitry:**
  - $R_f = 100k\Omega$
  - $i_{ia} = 0.01\, pA/\sqrt{Hz}$
  - $v_{ia} = 12\, nV/\sqrt{Hz}$

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*LF356 Op Amp Data Sheet*

**LF155/LF156/LF256/LF257/LF355/LF356/LF357**

**JFET Input Operational Amplifiers**

**General Description**

These are the first monolithic JFET input operational amplifiers to incorporate well matched, high voltage JFETs on the same chip with standard bipolar transistors (BSFET™ Technology). These amplifiers feature low input bias and offset currents/low offset voltage and offset drift coupled with offset adjust which does not degrade drift or common-mode rejection. The devices are also designed for high slew rate, wide bandwidth, extremely fast settling time, low voltage and current noise and low 1/f noise corner.

**Features**

- Replace expensive hybrid and module FET op amps
- Rugged JFETs allow blow-out free handling compared with MOSFET input devices
- Excellent for low noise applications using either high or low source impedance—very low 1/f corner
- Offset adjust does not degrade drift or common-mode rejection as in most monolithic amplifiers
- New output stage allows use of large capacitive loads (5,000 pF) without stability problems
- Internal compensation and large differential input voltage capability

**Applications**

- Precision high speed integrators
- Fast D/A and A/D converters
- High impedance buffers
- Wideband, low noise, low drift amplifiers

**Uncommon Features**

- Extremely fast settling time to 0.01%
- Fast slew rate
- Wide gain bandwidth
- Low input noise voltage

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*Example ARW Calculation*
Example ARW Calculation (cont)

Get rotation rate to output current scale factor:

\[
\frac{21\mu F}{21k\Omega \cdot \left(\frac{10k\Omega}{50k\Omega}\right) = 2.93 \times 10^{12}}
\]

\[
\Theta(j\omega) = \frac{j(10k\Omega)}{\left(\frac{10k\Omega}{50k\Omega}\right)^2 + \frac{j(10k\Omega)}{50k\Omega}} = \frac{j(10k\Omega)}{1.25 \times 10^{10} - j(10k\Omega)}
\]

\[
|\Theta(j\omega)| = \frac{30k}{\sqrt{1.25 \times 10^{10} + (10k\Omega)^2}} = 0.000024 \quad \text{pRs} \times 10^{-6} \text{F/m}
\]

\[
\frac{21\mu F}{21k\Omega} = \frac{\epsilon_{0}k_{l}w_{p}}{d} = \frac{2000\epsilon_{0}}{1.25 \times 10^{10} - j(10k\Omega)}
\]

Assuming electrode covers the whole sidewall:

Then, get noise:

\[
\frac{\tilde{\alpha}}{\text{av}} \text{en} = \frac{4\pi kT}{R_{x}} \frac{|\Theta(j\omega)|^2}{1 + \frac{4\pi kT}{R_{f}} + \frac{\epsilon_{0}k_{l}}{\Delta f} + \frac{\eta_{m}}{\Delta f}}
\]

Example ARW Calculation (cont)

\[
R_{x} = \frac{21\mu F \cdot (8.60 \times 10^{-9})}{(50k\Omega) \cdot (1.25 \times 10^{10})} = 110.6k\Omega
\]

\[
\frac{\tilde{\alpha}}{\text{en}} \text{eq}_{\text{tot}} = \frac{1.66 \times 10^{-26}}{(110.6k\Omega)} = 1.66 \times 10^{-26} \text{A/H}_{2}
\]

\[
\frac{\tilde{\alpha}}{\text{en}} \text{eq}_{\text{tot}} = \frac{1.66 \times 10^{-26}}{1.25 \times 10^{12}} = 1.66 \times 10^{-42} \text{A/H}_{2}
\]

\[
\Delta f_{\text{eq}} = \frac{1.66 \times 10^{-26}}{1.25 \times 10^{12}} = 1.66 \times 10^{-42} \text{A/H}_{2}
\]

\[
\Delta f_{\text{eq}} = \frac{1.66 \times 10^{-26}}{1.25 \times 10^{12}} = 1.66 \times 10^{-42} \text{A/H}_{2}
\]

And finally:

\[
\text{ARW} = \frac{1}{60} \Delta f_{\text{eq}} = \frac{1}{60} \left(9.44 \times 10^{-12}ight) = 1.57 \text{pRs} \times 10^{-12} \text{A/H}_{2}
\]
What if $\omega_d = \omega_s$?

If $\omega_d = \omega_s = 15 \text{kHz}$, then $|\mathcal{D}(\omega_d)|^2 = 1$ and

$$A = \frac{k_d}{k_s} Q_d Q_s \eta_d \eta_s |\mathcal{D}(\omega_d)|^2 = \frac{2 Q_d Q_s \eta_d \eta_s}{2(3000 \mu)(20 \mu)(2)(1000 \text{ e})} \approx 10^{-7} \text{ C}$$

$$\frac{\Delta \text{eq}_{\text{HT}}}{\Delta f} = \frac{(1.66 \times 10^{-10}) (1) + (1.66 \times 10^{-10}) (0.61^2) + (12 \mu)^2}{1.5 \times 10^{-26} \text{ A}/\text{Hz}^2} \approx 1.66 \times 10^{-26} \text{ A}/\text{Hz}^2$$

Now, the second element dominates!

$$\frac{\Delta \text{eq}_{\text{HT}}}{\Delta f} = 1.66 \times 10^{-26} \text{ A}/\text{Hz}^2 \quad \Rightarrow \quad \Delta \text{eq}_{\text{HT}} \approx 4 \times 10^{-12} \text{ A}/\text{Hz}^2$$

$$\sigma_{\text{HT}} = \frac{\Delta \text{eq}_{\text{HT}}}{\Delta f} = \frac{4 \times 10^{-12}}{10^{12}} \left( \frac{3600}{3600} \right) \approx 0.476 \text{ (\% Hz)}/\text{Hz}$$

And finally:

$$\text{ARV} = \frac{1}{60} \sigma_{\text{HT}} = \frac{1}{60} (0.476) \approx (0.0079 \% \text{ Hz}) = \text{ Navigation grade}$$