Lecture 14: Beam Combos II

Announcements:
- HW#4 online soon, due Tuesday, 3/17, at 8 a.m.
- Midterm Exam: Thursday, March 19, 9:30-11:00 a.m., 521 Cory (right here)
- UC Berkeley has stopped ground classes in an effort to suppress Coronavirus
- This is a video-recorded lecture, as will be subsequent lectures until the university goes back to ground classes
- Office hours are going to Zoom per my recent Piazza post
- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients
- Last Time:
  - Finished beam combinations and mechanical spring circuits, using a mechanical filter example
  - Now, continue with examples and methods for handling stress in beam combos

![Micromechanical Filter Diagram]
Get $k_b$:

$\frac{k_{cs}}{2}$

Series $\rightarrow$ \frac{k_{cs}}{2}$

Parallel $\rightarrow k_{cs}$

parallel $\rightarrow \frac{k_{cs}}{2} = k_b$

\[ \therefore k_A = k_c + k_{combined} \]

\[ k_c = \frac{24EI_s}{L^3} \]

\[ k_{cs} = \frac{24EI_s}{L^2} \]

\[ k_A = k_c + k_{cs} \]

Tensioned Spring (Non-Idenity)

* Important case for MEMS suspensions, since the thin films comprising them are often under residual stress
* Consider small deflection case: $y(x) \ll L$

Governing differential equation: (Euler Beam Equation)

\[ EI_s \frac{d^4 y}{dx^4} - S \frac{d^2 y}{dx^2} = F \delta(x - L) \]

Heuristic Derivation for the Euler Beam Equation:

Consider first a straight beam under an axial stress:

\[ \sigma_0 \]

When the beam is straight:

\[ \sigma_0 \]

... but when the beam bends:

Thin beam
Upward pressure $P_o$ to counteract the downward force from to keep everything in static equilibrium.

For ease of analysis:
Assume the beam bends to an angle $\theta$

Download vertical force: $250 WH$

Get upward force due to $P_o$:

$$P_2(\theta) = P_o \sin \theta$$

$$F_u = \int_0^\theta (P_2(\theta)W(R\theta)) = -P_o W R \cos \theta \bigg|_0^\theta$$

$$= 2RWP_o$$

[Equilibrium] $2RWP_o = 250 WH$ $\rightarrow$ $P_o = \frac{50 WH}{R}$

$q_o$: beam load/unit length = $P_o W$, $R = \frac{d^2w}{dx^2}$ beam displacement

$q_o$: $50 WH \frac{dw}{dx}$ generalizes to the case of small displacement of angles.

Using the Differential Beam Bending Eq

$$\frac{d^2w}{dx^2} = \frac{M}{EI} \quad ?? \quad \frac{dy}{dx} = \frac{q}{EI}$$

* Relationship between forces & moments on a Fully-Loaded Differential Beam Element

For total force:

$$q = \text{unit length}$$

$$\text{total force} = q dx + (V+dv) - V = 0$$

$$\Rightarrow \frac{dv}{dx} = -q$$

$$\Rightarrow \text{total moment} \text{car to left-hand edge} = 0$$

$$M = (M+\Delta M) - M - (V+dv)dx = \frac{1}{2} q dx^2 = 0$$

[Total Static Equilibrium] $\Rightarrow$ total force $= 0$

$$\text{neglect products of differentials}$$
Using (1) + (2):

\[
\frac{d^2 M}{dx^2} = \frac{dV}{dx} = -q \quad \text{ext. load}
\]

\[
EI \frac{d^2 y}{dx^2} = q + q_0 \quad \text{eq. w/ load from axial stress}
\]

\[q_0 = 60WH \frac{d^2 y}{dx^2} \]

\[
EI \frac{d^3 y}{dx^3} + (60WH) \frac{d^3 y}{dx^3} = q \quad \text{(free tension in beam)}
\]

\[
\text{Euler Beam Equation}
\]

**Compact-Guided Beam Under Axial Load**

* Important case for MEMS suspensions, since the thin films comprising them are often under residual stress
* Consider small deflection case: \( y(x) < L \)

Governing differential equation: (Euler Beam Equation)

\[
EI \frac{d^4 y}{dx^4} - S \frac{d^2 y}{dx^2} = F\delta(x-L)
\]

Axial Load

Unit impulse @ \( x=L \)

Need to solve this, then find the stiffness against this force (@ this location)

* Can solve the ODE using standard methods

  - Senturta, pp. 232-235: solves ODE for case of point load on a clamped-clamped beam (which defines B.C.'s)

* Result from Timoshenko: (Note: These include both bending & axial strain)

  - For \( S > 0 \) (tension)
    \[
k^{-1} = \frac{pL - 2 \tanh(pL/2)}{|p|S} = y(x = L)\quad \text{stiffness}
\]

  - For \( S < 0 \) (compression)
    \[
k^{-1} = \frac{-pL + 2 \tan(pL/2)}{|p|S} = y(x = L)
\]

where \( p = \frac{|S|}{\sqrt{EI}} \)
Folded Beams Are Not Perfect

1. This beam experiences tension.
2. Compress this beam.
3. Apply force on folding line.

- Shoulder expands
- Effective arch (like the structure is symmetric)
- Produkt mover with the substrate

Get $S$:

1. If the polymer structural material stren is $\varepsilon_r$, then the shoulder expands $\Delta L_s = \varepsilon_r L_s$
2. This then applies a load to the beams, $\Delta L = L_s$

- Beam Shear:
  \[ \varepsilon_b = \frac{\Delta L}{2L} = \frac{\Delta L_s}{2L} = \pm \varepsilon_r \frac{L_s}{2L} \]
  - Shear Force:
    \[ S = \pm E \varepsilon_r \left( \frac{L_s}{2L} \right) W_L \]
  - Axial tension

- Spring Constants:
  \[ k = \left( k_{com}^{-1} + k_{ten}^{-1} \right)^{-1} \]

- 

\[ k = 4 \left( \frac{-pL + 2 \tan(pL/2)}{p|S|} + \frac{pL - 2 \tanh(pL/2)}{p|S|} \right)^{-1} \]
Some problem as before: Take a beam, apply a force.

1. Apply force.

2. Beam responds by bending.

3. This force has done work:
   \[ W = F \cdot y(x) \]

   magnitude of "deformed by shape."

4. Strain generated.
   So the beam has received an influx of stored energy.

Then

\[ U = \text{Stored Energy} - \text{Work Done} \rightarrow 0 \]

When we choose the right shape.

This is how we get the beam's response to F!