Lecture 16: Resonance Frequency

- Announcements:
  - This is a video-recorded lecture, as will be subsequent lectures until the university goes back to ground classes
  - No new homework (so you can enjoy your Spring Break) ... but first ...
  - ... Midterm Exam: Remote Exam, Thursday, March 19, 9:30 a.m.-12:00 noon
    - See Piazza post for procedural changes
    - Main differences:
      - No need to print out the exam
      - We will use Zoom proctoring
- Reading: Senturia, Chpt. 10: §10.5, Chpt. 19
- Lecture Topics:
  - Estimating Resonance Frequency
  - Lumped Mass-Spring Approximation
  - ADXL-50 Resonance Frequency
  - Distributed Mass & Stiffness
  - Folded-Beam Resonator
  - Resonance Frequency Via Differential Equations
- Last Time:
  - Finished Lump Mass-Spring Approximation
  - Now, address distributed mass & stiffness ...

\[ \omega_0 = \sqrt{\frac{k}{m}} \Rightarrow \text{good for problems where mass of stiffness can be separated, i.e., they are distinct} \]

- The proof mass of the ADXL-50 is many times larger than the effective mass of its suspension beams
  - Can ignore the mass of the suspension beams (which greatly simplifies the analysis)
- Suspension Beam: \( L = 260 \, \mu m, h = 2.3 \, \mu m, W = 2 \, \mu m \)

In fabrication: purposefully introduce a tensile stress in the beams!

\[ \sigma = \frac{F}{A} \]

\( A \) large \( \Rightarrow \) why?

To avoid compression at all cost – buckling = dead device
Bending Contribution

$$k_b = k_{el} k_c \left( \frac{1}{k_c} + \frac{1}{k_c} \right)^{-1} = \frac{k_c}{2} = \frac{3 EI h^3}{12 L^3}$$

$$\Rightarrow k_b = EW \left( \frac{h}{L} \right)^3 = 0.24 \text{ N/m}$$

Stretching Contribution

$$F_g = \frac{5}{7} \frac{S \sin \theta}{\theta} \leq \frac{5}{L} \rho$$

Assume small

displacements

$$k_{st} \approx \frac{6}{L}$$

$$k_{st} = \frac{6}{L} = 0.88 \text{ N/m}$$

Get the total spring constant

$$k_{tot} = 4 (k_b + k_{st}) = 4 (0.24 + 0.88) = 4.5 \text{ N/m}$$

Now, get the resonance freq:

$$f_0 = \frac{1}{2 \pi} \sqrt{\frac{k}{M}} = \frac{1}{2 \pi} \sqrt{\frac{4.5 \text{ N/m}}{162 \times 10^{-12} \text{ kg}}} = 26.5 \text{ kHz}$$

Axial: 28

Detected: 24 kHz

Difference?

\(< \text{ Capacitive transducer} \rightarrow \text{ electrical stiffness}\)
Find the Resonance Frequency when Mass & Stiffness Are Distributed

* Vibrating structure displacement function:
  \[ y(x, t) = \ddot{y}(x) \cos(\omega t) \]

  Maximum displacement function (i.e., mode shape function) Seen when velocity \( \dot{y}(x, t) = 0 \)

* Procedure for determining resonance frequency:
  1. Use the static displacement of the structure as a trial function and find the strain energy \( W_{\text{max}} \) at the point of maximum displacement (e.g., when \( t=0, \pi/\omega, \ldots \))
  2. Determine the maximum kinetic energy when the beam is at zero displacement (e.g., when it experiences its maximum velocity)
  3. Equate energies and solve for frequency

Get Maximum Kinetic Energy

Velocity:
\[ v(x,t) = \ddot{y}(x,t) = -\omega \dot{y}(x) \sin(\omega t) \]

Maximum Kinetic Energy, \( K_{\text{max}} \):
\[
K_{\text{max}} = \frac{1}{2} \rho W \int v^2(x,t) \, dx = \frac{1}{2} \rho W \int [\dot{y}(x)]^2 \, dx
\]

To get frequency: \( \omega = \sqrt{\frac{K_{\text{max}}}{W}} \)

- \( \omega \) = radian resonance freq.
- \( W_{\text{max}} \) = maximum potential energy
- \( \rho \) = density of the structural material
- \( W \) = beam width
- \( h \) = " thickness
- \( \dot{y}(x) \) = resonance mode shape
- Derive an expression for the resonance frequency of the above structure

**Approximation**

\[ m = \text{shuttle mass} \]
\[ k = k_c \]

\[ \omega_0 = \sqrt{\frac{k}{m}} \]

But not accurate enough for some applications.

- For better accuracy, must integrate

**Use the Rayleigh–Ritz Method: (energy method)**

\[ \kappa_{\text{max}}: W_{\text{max}} = \frac{1}{2} k x^2 \]

Find the kinetic energy, one piece at a time

\[ \kappa_{\text{max}} = k_s + k_t + k_b \]

- shuttle
- truss beams

\[ = \frac{1}{2} k_s M_s + \frac{1}{2} k_t M_t + \frac{1}{2} \int k_b \, dM_b \]

Velocity of the Shuttle: \( N_s = \omega_0 x_0 \)

\[ \kappa_s = \frac{1}{2} k_s M_s = \frac{1}{2} \omega_0^2 x_0^2 M_s = \kappa_s \]

Velocity of Truss: \( N_t = \frac{1}{2} N_s = \frac{1}{2} \omega_0 x_0 \)

\[ \kappa_t = \frac{1}{2} (\frac{1}{2} \omega_0 x_0)^2 M_t = \frac{1}{8} \omega_0^2 x_0^2 M_t = \kappa_t \]

mass of both trusses
Velocity of the Beam Segments: $\text{first beam } [AB]$

\[ \frac{x_0}{2} \]

\[ \text{Guided B.C.} \]

\[ \text{Fixed B.C.} \]

\[ \text{Need the mode shape.} \]

Assume the mode shape is the same as the static displacement shape.

\[ \text{Segment } [AB]: \]

\[ \hat{x}(y) = \frac{F_x}{4EI_2} \left( 3y^2 - 2y^3 \right), \quad 0 \leq y \leq 1 \]  

At $y=1$: $x(L) = \frac{x_0}{2}$, $\frac{F_xL^3}{4EI_2} = 8 \text{ c.}$

Substitute into (1):

\[ x(y) = \frac{x_0}{2} \left[ 3 \left( \frac{y}{2} \right)^2 - 2 \left( \frac{y}{2} \right)^3 \right] \]

Which yields for velocity:

\[ v_0(x)[y]_{[AB]} = \frac{x_0}{2} \left[ 3 \left( \frac{y}{2} \right)^2 - 2 \left( \frac{y}{2} \right)^3 \right] \omega_0 \]

Plugging into expression for $K_{[AB]}$:

\[ K_{[AB]} = \frac{1}{2} \int_0^L \frac{x_0^2 \omega_0^2}{4} \left[ 3 \left( \frac{y}{2} \right)^2 - 2 \left( \frac{y}{2} \right)^3 \right]^2 dy \]

\[ = \frac{x_0^2 \omega_0^2 M_{[AB]}}{8L} \int_0^L \left[ 3 \left( \frac{y}{2} \right)^2 - 2 \left( \frac{y}{2} \right)^3 \right]^2 dy \]

\[ M_{[AB]} = \text{static mass} \]

\[ K_{[AB]} = \frac{13}{280} x_0^2 \omega_0^2 M_{[AB]} \]

For segment $[CD]$:

\[ \hat{v}(y)[y]_{[CD]} = x_0 \left[ 1 - \frac{3}{2} \left( \frac{y}{L} \right)^2 + \left( \frac{y}{L} \right)^3 \right] \omega_0 \]

Thus:

\[ K_{[CD]} = \frac{x_0^2 \omega_0^2 M_{[CD]}}{2L} \int_0^L \left[ 1 - \frac{3}{2} \left( \frac{y}{L} \right)^2 + \left( \frac{y}{L} \right)^3 \right]^2 dy \]

\[ K_{[CD]} = \frac{93}{280} x_0^2 \omega_0^2 M_{[CD]} \]

Let $M_b = \text{total mass of all } \text{static mass of } \text{beam } [CP]$.

Thus:

\[ K_4 = 4K_{[AB]} + 4K_{[CD]} = \frac{6}{35} x_0^2 \omega_0^2 M_b \]
\[
K_{\text{max}} = \frac{x_0^2}{\omega_0^2} \left[ \frac{1}{2} M_s + \frac{1}{6} M_t + \frac{6}{35} M_b \right]
\]

for the total mechanical

\[
W_{\text{max}} = \text{max. potential energy} = \text{equal to the work done to achieve maximum deflection}
\]

\[
W_{\text{max}} = \frac{1}{2} k_x x_0^2
\]

Then, using Rayleigh–Ritz:

\[
K_{\text{max}} = W_{\text{max}}
\]

\[
\frac{x_0^2}{\omega_0^2} \left[ \frac{1}{2} M_s + \frac{1}{6} M_t + \frac{6}{35} M_b \right] = \frac{1}{2} k_x x_0^2
\]

\[
\omega_0 = \left[ \frac{k_c}{M_{\text{eq}}} \right]^{1/2}
\]

where \( M_{\text{eq}} = M_s + \frac{1}{4} M_t + \frac{12}{25} M_b \)

(Resonance Freq. of a Folded-Beam)

Suspended Shuttle}