Lecture Outline

* Reading: Senturia, Chpt. 14, Chpt. 16, Chpt. 21
* Lecture Topics:
  - Gyroscopes
  - Gyro Circuit Modeling
  - Minimum Detectable Signal (MDS)
    - Noise
    - Angle Random Walk (ARW)
Gyroscopes

Classic Spinning Gyroscope

- A gyroscope measures rotation rate, which then gives orientation → very important, of course, for navigation
- Principle of operation based on conservation of momentum
- Example: classic spinning gyroscope

Rotor will preserve its angular momentum (i.e., will maintain its axis of spin) despite rotation of its gimbled chassis
Vibratory Gyroscopes

- Generate momentum by vibrating structures
- Again, conservation of momentum leads to mechanisms for measuring rotation rate and orientation
- Example: vibrating mass in a rotating frame

\[
\begin{align*}
\text{Mass at rest} & \quad y^* \\
\text{Driven into vibration along the } y\text{-axis} & \\
\text{y-displaced mass} & \\
\text{Capacitance between mass and frame } = \text{constant}
\end{align*}
\]

Basic Vibratory Gyroscope Operation

- Principle of Operation:
  - Tuning Fork Gyroscope:

\[
\begin{align*}
\text{Driven Vibration} & \quad z \\
\text{Coriolis (Sense) Response} & \\
\text{Coriolis Torque} & \\
\text{Input Rotation} & \quad \Omega
\end{align*}
\]

\[
\begin{align*}
\text{Detect motion out-of-the plane of the tuning fork as rotation!}
\end{align*}
\]
**Basic Vibratory Gyroscope Operation**

**Principle of Operation**
- Tuning Fork Gyroscope:

  ![Diagram of Tuning Fork Gyroscope]

  - Input Rotation
  - Driven Vibration @ \( f_o \)
  - Coriolis (Sense) Response
  - Coriolis Torque

**Drive/Sense Response Spectra**
- Amplitude vs. Frequency
- Drive Response vs. Sense Response

\[
\ddot{a_c} = 2\dot{v} \times \overline{\Omega}
\]

**Vibratory Gyroscope Performance**

**Principle of Operation**
- Tuning Fork Gyroscope:

  ![Diagram of Vibratory Gyroscope]

  - Input Rotation
  - Driven Vibration @ \( f_c \)
  - Coriolis (Sense) Response
  - Coriolis Torque

**Equations**
- Beam Mass
- Beam Stiffness
- Sense Frequency (in same direction)

\[
\ddot{x} = \frac{\ddot{F_c}}{k} = \frac{m\ddot{a}_c}{k} = \frac{\ddot{a}_c}{\omega_c^2}
\]

**To maximize the output signal \( x \), need:**
- Large sense-axis mass
- Small sense-axis stiffness
- (Above together mean low resonance frequency)
- Large drive amplitude for large driven velocity (so use comb-drive)
- If can match drive freq. to sense freq., then can amplify output by \( Q \) times
MEMS-Based Gyroscopes

Vibrating Ring Gyroscope

Najafi, Michigan

Laser
Polarizer
Rb/Xe Cell
Photodiode

3.2 mm
1 mm

Tuning Fork Gyroscope

Draper Labs.

Nuclear Magnetic Resonance Gyro [NIST]

MEMS-Based Tuning Fork Gyroscope

• In-plane drive and sense modes pick up z-axis rotations
• Mode-matching for maximum output sensitivity
• From [Zaman, Ayazi, et al, MEMS'06]
**MEMS-Based Tuning Fork Gyroscope**

- Drive and sense axes must be stable or at least track one another to avoid output drift.

**Problem:** if drive frequency changes relative to sense frequency, output changes ⇔ bias drift.

**Need:** small or matched drive and sense axis temperature coefficients to suppress drift.
**Mode Matching for Higher Resolution**

- For higher resolution, can try to match drive and sense axis resonance frequencies and benefit from Q amplification.

![Diagram showing mode matching](image)

- Problem: mismatch between drive and sense frequencies → even larger drift!

**Need**: small or matched drive and sense axis temperature coefficients to make this work.

**Issue: Zero Rate Bias Error**

- Imbalances in the system can lead to zero rate bias error.

![Diagram showing zero rate bias error](image)

- Drive imbalance → off-axis motion of the proof mass
- Mass imbalance → off-axis motion of the proof mass
- Quadrature output signal that can be confused with the Coriolis acceleration
- Output signal in phase with the Coriolis acceleration
Nuclear Magnetic Res. Gyroscope

- The ultimate in miniaturized spinning gyroscopes?
- From CSAC, we may now have the technology to do this

Better if this is a noble gas nucleus (rather than e-), since nuclei are heavier ⇒ less susceptible to B field

Soln: Spin polarize Xe^{129} nuclei by first polarizing e- of Rb^{87} (a la CSAC), then allowing spin exchange

Challenge: suppressing the effects of B field

MEMS-Based Tuning Fork Gyroscope

[Zaman, Ayazi, et al, MEMS'06]

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Determining Sensor Resolution

MEMS-Based Tuning Fork Gyroscope

[Zaman, Ayazi, et al., MEMS'06]
**Drive Axis Equivalent Circuit**

- Generates drive displacement velocity $x_d$ to which the Coriolis force is proportional.

**Drive-to-Sense Transfer Function**

- Drive Mode: $\dot{x}_d = \omega_d x_d$
- Sense Mode: $\dot{x}_s = \omega_s x_s$

Rotation-Induced Coriolis Force:

$$a_s = 2\omega_x x_d \Omega \sin 90^\circ$$

Where $\Omega$ is the angular velocity, $\omega_x$ is the angular frequency, and $x_d$ is the drive displacement.
Gyro Readout Equivalent Circuit
(for a single tine)

\[ F_c = m \ddot{a}_c = m \cdot (2 \dddot{x}_d \times \Omega) \]

- Noise Sources
- Gyro Sense Element
- Output Circuit
- Signal Conditioning Circuit (Transresistance Amplifier)

* Easiest to analyze if all noise sources are summed at a common node

Minimum Detectable Signal (MDS)

* Minimum Detectable Signal (MDS): Input signal level when the signal-to-noise ratio (SNR) is equal to unity

* The sensor scale factor is governed by the sensor type
* The effect of noise is best determined via analysis of the equivalent circuit for the system
**Move Noise Sources to a Common Point**

- Move noise sources so that all sum at the input to the amplifier circuit (i.e., at the output of the sense element).
- Then, can compare the output of the sensed signal directly to the noise at this node to get the MDS.

![Diagram of Noise Source Summation](image)

**Gyro Readout Equivalent Circuit**

(for a single tine)

- Easiest to analyze if all noise sources are summed at a common node.

![Gyro Readout Circuit](image)
Noise Sources

\[ \dot{F}_{c} = m\ddot{a}_{c} = m \cdot (2\dot{x}_{d} \times \vec{\Omega}) \]

\[ i_{o} = \eta_{e} \cdot 1 \]

\[ v_{eq}^{2} \]

\[ i_{eq}^{2} \]

\[ C_{p} \]

\[ R_{f} \]

\[ V_{0} \]

Gyro Sense Element

Output Circuit

Signal Conditioning Circuit

(Transresistance Amplifier)

* Here, \( v_{eq}^{2} \) and \( i_{eq}^{2} \) are equivalent input-referred voltage and current noise sources.

Noise
Noise

- Noise: Random fluctuation of a given parameter \( I(t) \)
- In addition, a noise waveform has a zero average value

\[ \text{Avg. value (e.g., could be DC current)} \]

- We can't handle noise at instantaneous times
- But we can handle some of the averaged effects of random fluctuations by giving noise a power spectral density representation
- Thus, represent noise by its mean-square value:

\[
\text{Let } i(t) = I(t) - I_D \\
\text{Then } \overline{i^2} = \overline{(I - I_D)^2} = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} |I - I_D|^2 \, dt
\]

Noise Spectral Density

- We can plot the spectral density of this mean-square value:

\[
\frac{\overline{i^2}}{\Delta f} \quad \text{[units}^2/\text{Hz}]\]

One-sided spectral density
\( \rightarrow \) used in circuits \( \rightarrow \) measured by spectrum analyzers

Two-sided spectral density
(1/2 the one-sided)

Often used in systems courses

\[ \overline{i^2} = \text{integrated mean-square noise spectral density over all frequencies (area under the curve)} \]
Circuit Noise Calculations

Inputs
- Deterministic: $v_i(j\omega)$
- Linear Time-Invariant System: $H(j\omega)$
- Random: $S_i(\omega)$

Outputs
- Deterministic: $v_o(j\omega) = H(j\omega)v_i(j\omega)$
- Random: $S_o(\omega) = \left|H(j\omega)\right|^2 S_i(\omega)$

**Mean square spectral density**

**Root mean square amplitudes**

Handling Noise Deterministically

* Can do this for noise in a tiny bandwidth (e.g., 1 Hz)

Can approximate this by a sinusoidal voltage generator (especially for small $B$, say 1 Hz)

Why? Neither the amplitude nor the phase of a signal can change appreciably within a time period $1/B$.
Systematic Noise Calculation Procedure

1. For $i_{n1}^2$, replace w/ a deterministic source of value $i_{n1} = \sqrt{\frac{i_{n1}^2}{\Delta f}}$, (1 Hz)

2. Calculate $v_{on1}(\omega) = i_{n1}(\omega)H(j\omega)$ (treating it like a deterministic signal)

3. Determine $v_{on1}^2 = \frac{i_{n1}^2}{\Delta f} \cdot |H(j\omega)|^2$

4. Repeat for each noise source: $\frac{i_{n1}^2}{\Delta f}, v_{n2}, v_{n3}$

5. Add noise power (mean square values)

$$\bar{v}_{onTOT}^2 = v_{on1}^2 + v_{on2}^2 + v_{on3}^2 + v_{on4}^2 + \cdots$$

$$v_{onTOT} = \sqrt{\bar{v}_{onTOT}^2} = \sqrt{v_{on1}^2 + v_{on2}^2 + v_{on3}^2 + v_{on4}^2 + \cdots}$$

Total rms value
Determining Sensor Resolution

Example: Gyro MDS Calculation

\[ \vec{F}_c = m \vec{a}_c = m \cdot (2 \vec{x}_d \times \vec{\Omega}) \]

- The gyro sense presents a large effective source impedance
- Currents are the important variable; voltages are “opened” out
- Must compare \( i_o \) with the total current noise \( i_{eqTOT} \) going into the amplifier circuit
Example: Gyro MDS Calculation (cont)

\[ F_c = m\ddot{a}_e = m \cdot (2\dot{x}_d \times \Omega) \]

\[ v_{eq} = \frac{R_f}{C_p} i_o \]

* First, find the rotation to \( i_o \) transfer function:

\[ \dot{\chi}_s = \frac{\omega_0}{k_s} \Theta_s(j\omega) \]

\[ F_c = F_e = 2\omega d x_m \Omega_m \]

\[ \dot{\chi}_s = 2 \frac{\omega d}{\omega_s} Q x_d \Theta(j\omega) \]

Example: Gyro MDS Calculation (cont)

\[ i_o \cdot \eta_s \dot{\chi}_s = 2 \frac{\omega d}{\omega_s} Q x_d \eta_e \Theta(j\omega) \cdot S \]

Where \( A \) = scale factor

When \( \Omega_s \Omega_m = MDS \), \( i_o = i_{eq\text{ort}} \) input-referred noise current entering the sense amplifier - in pA/\( \sqrt{Hz} \)

\[ \dot{i}_{eq\text{ort}} = AS_{min} \]

\[ S_{min} = \frac{i_{eq\text{ort}}}{A} \left( \frac{3600s}{hr} \right) \left( \frac{180}{\pi} \right) \left[ \frac{(\%\text{hr})}{\text{Hz}} \right] \]

\[ \text{Angle Random Walk} = ARW = \frac{1}{60} S_{min} \left[ \frac{\%\text{hr}}{\text{Hz}} \right] \]

Easier to determine directional error as a function of elapsed time.
Example: Gyro MDS Calculation (cont)

\[ \dot{\vec{F}}_c = m\ddot{\vec{a}}_e = m \cdot (2\dot{\vec{x}}_d \times \vec{\Omega}) \]

Now, find the \( i_{eqTOT} \) entering the amplifier input:

\[ i_{eqTOT} = i_s + i_{eq} \quad \Rightarrow \quad i_{eqTOT} = \ddot{i}_s + \dot{i}_{eq} + \ddot{i}_{eq} + \dot{i}_{eq} + \frac{N_a}{R_f} \frac{\ddot{v}_{eq}}{\dot{f}} \]

Browin motion noise of the sense element determined entirely by the noise in \( r_x \rightarrow \frac{\ddot{v}_{eq}}{\dot{f}} \)

\( f_{eq} = \frac{YK}{R_x} \)

\( R_s \): large \( \Rightarrow \frac{N_{eq}}{i_{eq}} \) "opened" out

Example: Gyro MDS Calculation (cont)

\[ L_x \quad C_x \quad R_x \rightarrow \text{To Amplifier Input} \]

\[ N_{eqTOT} = \frac{4kT}{R_x} \]

\[ \Rightarrow \quad \frac{\ddot{i}_s}{\dot{f}} = \frac{YkT}{R_x} (i(j\omega)) \]

Thus:

\[ \frac{\ddot{i}_s}{\dot{f}} = \frac{YK}{R_x} |(j\omega)|^2 + \frac{N_a}{R_f} \frac{\ddot{v}_{eq}}{\dot{f}} \]

Learn to get these from EE240.

\( \ddot{i}_s \) or just get them from a data sheet...
Example ARW Calculation

- **Example Design:**
  - Sensor Element:
    - \( m = (100 \text{µm})(100 \text{µm})(20 \text{µm})(2300 \text{kg/m}^3) = 4.6 \times 10^{-10} \text{kg} \)
    - \( \omega_s = 2\pi (15 \text{kHz}) \)
    - \( \omega_d = 2\pi (10 \text{kHz}) \)
    - \( k_s = \omega_s^2 m = 4.09 \text{ N/m} \)
    - \( x_d = 20 \text{ µm} \)
    - \( Q_s = 50,000 \)
    - \( V_p = 5 \text{V} \)
    - \( h = 20 \text{ µm} \)
    - \( d = 1 \text{ µm} \)

- Sensing Circuitry:
  - \( R_f = 100k\Omega \)
  - \( i_{ia} = 0.01 \text{ pA/√Hz} \)
  - \( v_{ia} = 12 \text{nV/√Hz} \)
Example ARW Calculation (cont)

Get rotation rate to output current scale factor:

\[ A = \frac{Ld}{\omega_s} \left[ Q_s^2 \right] \frac{j(\omega_s)/(\omega_s^2 + \omega_s^2)}{ \omega_s^2 + \frac{j\omega_s \omega_s}{\omega_s^2 + \omega_s^2} } = \frac{j(10k)/(10k)^2}{(10k)^2 - (10k)^2 + j(10k)/(10k)} \frac{1}{1.25 \times 10^{-6} \text{j}(2k)} \]

\[ \Theta(j\omega) = \frac{\omega_s \omega_s}{\omega_s^2 + \omega_s^2} = \frac{j(3k)}{(1.25 \times 10^{-6})^2 + (3k)^2} = 0.000024 \]

Then, get noise:

\[ \frac{\Delta x}{\Delta f} = \frac{4kT}{R_f} \left| \Theta(j\omega) \right|^2 + \frac{4kT}{R_f} + \frac{\lambda}{\Delta f} + \frac{\lambda}{\Delta f} \left( \frac{1}{R_f} \right) \]

Example ARW Calculation (cont)

\[ R_X = \frac{L_{\text{mem}}}{Q_{\text{mem}}} = \frac{2\pi(5k)(4.6 \times 10^{-19})}{(5k)(8.5 \times 10^{-13})} = 110.6 \text{k}\Omega \]

\[ \frac{\Delta x}{\Delta f} = \frac{(1.66 \times 10^{-19})^2}{(110.6\Omega)} + \frac{(1.66 \times 10^{-19})^2}{(110.6\Omega)^2} + \frac{0.03^2}{(110.6\Omega)^2} + \frac{0.03^2}{(110.6\Omega)^2} \]

\[ \frac{\Delta x}{\Delta f} = 8.64 \times 10^{-27} \text{A}^2/\text{Hz} \]

Senser element noise insignificant.

Noise from \( R_f \) dominates.

\[ \frac{\Delta x}{\Delta f} = 1.64 \times 10^{-26} \text{A}^2/\text{Hz} \rightarrow \Lambda = \frac{1.64 \times 10^{-26}}{1.30 \times 10^{-12}} \]

\[ \Lambda_{\text{eq}} = 1.25 \times 10^{-12} \text{A}^2/\text{Hz} \]

And finally:

\[ \text{ARW} = \frac{1}{60} \quad \text{and} \quad \text{ARW} = \frac{1}{60} \times (9448) = 157 \% \text{hr} = \text{ARW} \]

Almost thermally dominated in 1 hour.
What if $\omega_d = \omega_s$?

If $\omega_d \omega_s = 15 \text{kHz}$, then $\left| \Theta (\omega_d) \right| \approx 1$ and

$$A = \frac{\omega_d}{\omega_s} \frac{Q_s}{Q_d} \frac{V_e}{V_d} \Theta (\omega_d) \approx 2 \frac{Q_s}{Q_d} \frac{V_e}{V_d} \Theta (\omega_d) \approx 2 \frac{2.5 \times 10^7}{2 \times 10^7} \left( \frac{1000 \text{eV}}{1000 \text{eV}} \right) \approx 1.77 \times 10^{-7} \text{C}$$

$$\frac{4 \Theta}{df} \approx \frac{(1.66 \times 10^{-10}) (1.26 \times 10^{-10})}{(110 \text{kHz})^{1/2}} \approx \frac{1.66 \times 10^{-10}}{1.10 \times 10^{-10}} \approx 1.51 \times 10^{-10} \frac{\text{m}}{\text{Hz}}$$

Now, the second element dominates!

$$\lambda_{eq} = \frac{\lambda}{4 \Theta} \approx \frac{1.66 \times 10^{-10}}{4 \times 10^{-13}} \approx 4.16 \times 10^{-1} \text{A/VHz}$$

And finally:

$$\text{ARV} = \frac{1}{60} \times \text{S}_{\text{min}} \approx \frac{1}{60} \times (0.476) \approx 0.0079 \% / \text{Hz}$$

$$= \text{Navigation grade!}$$