Lecture Outline

- Reading: Senturia Chpt. 16
- Lecture Topics:
  - Minimum Detectable Signal
  - Noise
    - Circuit Noise Calculations
    - Noise Sources
    - Equivalent Input-Referred Noise
  - Gyro MDS
    - Equivalent Noise Circuit
    - Example ARW Determination
Determining Sensor Resolution

Minimum Detectable Signal (MDS)

- Minimum Detectable Signal (MDS): Input signal level when the signal-to-noise ratio (SNR) is equal to unity

- The sensor scale factor is governed by the sensor type
- The effect of noise is best determined via analysis of the equivalent circuit for the system
Noise

- **Noise**: Random fluctuation of a given parameter $I(t)$
- In addition, a noise waveform has a zero average value

- We can't handle noise at instantaneous times
- But we can handle some of the averaged effects of random fluctuations by giving noise a power spectral density representation

- Thus, represent noise by its mean-square value:

Let $i(t) = I(t) - I_D$

Then $i^2 = (I - I_D)^2 = \lim_{T \to \infty} \frac{1}{T} \int_0^T |I - I_D|^2 dt$
Noise Spectral Density

We can plot the spectral density of this mean-square value:

\[ \overline{i^2} \quad [{\text{units}^2/\text{Hz}}] \]

- **One-sided spectral density**
  - used in circuits
  - measured by spectrum analyzers

- **Two-sided spectral density** (1/2 the one-sided)
  - Often used in systems courses

\[ \overline{i^2} = \text{integrated mean-square noise spectral density over all frequencies (area under the curve)} \]

Circuit Noise Calculations

- **Deterministic:**
  \[ v_o(j\omega) = H(j\omega)v_i(j\omega) \]

- **Random:**
  \[ S_o(\omega) = |H(j\omega)H^*(j\omega)|S_i(\omega) = |H(j\omega)|^2S_i(\omega) \]

\[ \sqrt{S_o(\omega)} = |H(j\omega)|\sqrt{S_i(\omega)} \quad \text{How is it we can do this?} \]
**Handling Noise Deterministically**

- Can do this for noise in a tiny bandwidth (e.g., 1 Hz)

\[ \frac{v_{n1}^2}{\Delta f} = S_1(f) \rightarrow v_{n1} = \sqrt{S_1(f) \cdot B} \]

Can approximate this by a sinusoidal voltage generator (especially for small B, say 1 Hz)

- Neither the amplitude nor the phase of a signal can change appreciably within a time period 1/B.

[This is actually the principle by which oscillators work → oscillators are just noise going through a tiny bandwidth filter]

**Systematic Noise Calculation Procedure**

- Assume noise sources are uncorrelated

1. For \( \frac{i_{n1}^2}{\Delta f} \), replace with a deterministic source of value

\[ i_{n1} = \sqrt{\frac{i_{n1}^2}{\Delta f}} \cdot (1 \text{ Hz}) \]
Systematic Noise Calculation Procedure

2. Calculate \( v_{on1}(\omega) = i_{n1}(\omega)H(j\omega) \) (treating it like a deterministic signal)

3. Determine \( \overline{v^2_{on1}} = \overline{i^2_{n1}} \cdot |H(j\omega)|^2 \)

4. Repeat for each noise source: \( \overline{i^2_{n1}}, \overline{v^2_{n2}}, \overline{v^2_{n3}} \)

5. Add noise power (mean square values)

\[
\overline{v^2_{onTOT}} = \overline{v^2_{on1}} + \overline{v^2_{on2}} + \overline{v^2_{on3}} + \overline{v^2_{on4}} + \cdots
\]

\[
V_{onTOT} = \sqrt{\overline{v^2_{on1}} + \overline{v^2_{on2}} + \overline{v^2_{on3}} + \overline{v^2_{on4}} + \cdots}
\]

Total rms value

Noise Sources
Thermal Noise

- Thermal Noise in Electronics: (Johnson noise, Nyquist noise)
  - Produced as a result of the thermally excited random motion of free e\(^-\)s in a conducting medium
  - Path of e\(^-\)s randomly oriented due to collisions

- Thermal Noise in Mechanics: (Brownian motion noise)
  - Thermal noise is associated with all dissipative processes that couple to the thermal domain
  - Any damping generates thermal noise, including gas damping, internal losses, etc.

- Properties:
  - Thermal noise is white (i.e., constant w/ frequency)
  - Proportional to temperature
  - Not associated with current
  - Present in any real physical resistor

\[
\frac{v^2_R}{R} = \frac{4kT}{\Delta f} \quad \text{or} \quad \frac{i^2_R}{R} = \frac{4kTR}{\Delta f}
\]

Note: These are one-sided mean-square spectral densities! To make them 2-sided, must divide by 2.

\[4kT = 1.66 \times 10^{-20} V \cdot C\]

Circuit Representation of Thermal Noise
Noise in Capacitors and Inductors?

- Resistors generate thermal noise
- Capacitors and inductors are noiseless → why?

\[ v(t) \]

Can oscillate forever

- Now, add a resistor:

\[ v(t) \]

Decays to zero

But this violates the laws of thermodynamics, which require that things be in constant motion at finite temperature

Need to add a forcing function, like a noise voltage \( v_R \) to keep the motion going → and this noise source is associated with \( R \)

Why 4kTR?

- Why is \( v_R = 4kTR\Delta f \) (a heuristic argument)
- The Equipartition Theorem of Statistical Thermodynamics says that there is a mean energy \((1/2)kT\) associated with each degree of freedom in a given system
- An electronic circuit possesses two degrees of freedom:
  - Current, \( i \), and voltage, \( v \)
  - Thus, we can write:

\[
\begin{align*}
\frac{1}{2}Li^2 &= \frac{1}{2}k_BT, \\
\frac{1}{2}Cv^2 &= \frac{1}{2}k_BT
\end{align*}
\]

- Similar expressions can be written for mechanical systems
  - For example: for displacement, \( x \)

\[
\frac{1}{2}kx^2 = \frac{1}{2}k_BT
\]
Why 4kTR? (cont)

- Why is $V_{th}^2 = 4kT\Delta f$? (a heuristic argument)
- Consider an RC circuit:

\[
\begin{align*}
E &= \frac{1}{2} kT = \frac{1}{2} CV^2_{th} \\
\therefore \ n_c^2 &= \frac{kT}{C} \quad \text{(total mean square voltage integrated over all freqs)} \\
\text{Question: What value of } \frac{\bar{V}_R^2}{\Delta f} \text{ (assuming white noise) gives us this?}
\end{align*}
\]

\[
\begin{align*}
\bar{V}_R^2 &= \int_0^\infty \left| \frac{1}{1+j\omega RC} \right|^2 \frac{
\bar{V}_R^2}{\Delta f} \, df \\
\text{[noise is white]} \quad \Rightarrow \quad \omega_b \left( \frac{1}{RC} \right) \\
\int \left[ \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) \right] \\
= \frac{1}{2\pi} \left. \frac{N_b\Delta f}{\omega_b} \tan^{-1} \left( \frac{\omega_b}{\omega_0} \right) \right|_0^\infty = \frac{1}{2\pi} \frac{N_b^2}{\Delta f} \left( \frac{\pi}{2} \omega_b - 0 \right) \\
= \frac{1}{4} \frac{\omega_b N_b^2}{\Delta f} \Rightarrow \frac{\bar{V}_R^2}{\Delta f} = \frac{4kT}{C} \Rightarrow \bar{V}_C^2 = 4kTR
\end{align*}
\]

\[\text{\underline{UC Berkeley}}\]
**Shot Noise**

- Associated with direct current flow in diodes and bipolar junction transistors
- Arises from the random nature by which e\(^{-}\)'s and h\(^{+}\)'s surmount the potential barrier at a pn junction
- The DC current in a forward-biased diode is composed of h\(^{+}\)'s from the p-region and e\(^{-}\)'s from the n-region that have sufficient energy to overcome the potential barrier at the junction
  → noise process should be proportional to DC current
- Attributes:
  - Related to DC current over a barrier
  - Independent of temperature
  - White (i.e., const. w/ frequency)
  - Noise power ~ \(I_D\) & bandwidth

\[
\frac{i_n^2}{\Delta f} = 2qI_D
\]

Charge on an e\(^{-}\) (=1.6×10\(^{-19}\)C)

**Flicker (1/f) Noise**

- In general, associated w/ random trapping & release of carriers from "slow" states
- Time constant associated with this process gives rise to a noise signal w/ energy concentrated at low frequencies
- Often, get a mean-square noise spectral density that looks like this:

\[
\frac{i_n^2}{\Delta f} \sim \frac{1}{f}
\]

\[
\frac{i_n^2}{\Delta f} = 2qI_D + K \left(\frac{I_D^a}{f^b}\right)
\]

- \(I_D = \) DC current
- \(K = \) const. for a particular device
- \(a = 0.5 \rightarrow 2\)
- \(b \sim 1\)
Example: Typical Noise Numbers

* Hookup the circuit below and make some measurements

\[
\begin{align*}
&\text{Low Noise Amplifier} \\
&\text{Measure w/ AC voltmeter} \\
&\text{Measure w/ spectrum analyzer} \\
&\text{Get Gaussian amplitude distribution} \\
&\text{68.2% within ±σ} \\
&\text{99.7% within ±3σ}
\end{align*}
\]

\[
\begin{align*}
&\text{AC Voltmeter} \\
&\sqrt{N_0^2} = (100)(64\mu V \text{ rms}) \\
&= 6.4\mu V \text{ rms}
\end{align*}
\]

\[
\begin{align*}
&\text{Spectrum Analyzer} \\
&\frac{1}{(2\pi)(1k\Omega)} = 60\text{ MHz} \\
&\text{20 dB/dec one-sided spectral density}
\end{align*}
\]
Back to Determining Sensor Resolution

MEMS-Based Tuning Fork Gyroscope

[Zaman, Ayazi, et al, MEMS'06]

Drive Voltage Signal

Drive Oscillation Sustaining Amplifier

Differential TransR Sense Amplifier

(-) Sense Output Current

(+) Sense Output Current

From Sense

From Sense

From Sense

From Sense

[EEC247B/MEC218: Introduction to MEMS Design LecM 17 C. Nguyen 11/18/08]
**Drive Axis Equivalent Circuit**

- Generates drive displacement velocity $\dot{x}_d$ to which the Coriolis force is proportional
- To Sense Amplifier (for synchronization)

**Drive-to-Sense Transfer Function**

Rotation-Induced Coriolis Force:

$$\ddot{a}_s = 2\omega_s \dot{x}_d \times \vec{\Omega}$$

Acts in the sense mode direction

$$a_s = 2\omega_s \dot{x}_d \Omega \sin 90^\circ$$

$$a_s = 2\omega_s \dot{x}_d \Omega$$
**Gyro Readout Equivalent Circuit**
(for a single tine)

\[ \ddot{F}_c = m\ddot{a}_c = m \cdot (2\ddot{x}_d \times \ddot{\Omega}) \]

- Easiest to analyze if all noise sources are summed at a common node

**Noise Sources**
- Gyro Sense Element
- Output Circuit
- Signal Conditioning Circuit (Transresistance Amplifier)

**Minimum Detectable Signal (MDS)**

- Minimum Detectable Signal (MDS): Input signal level when the signal-to-noise ratio (SNR) is equal to unity

- The sensor scale factor is governed by the sensor type
- The effect of noise is best determined via analysis of the equivalent circuit for the system
Move Noise Sources to a Common Point

- Move noise sources so that all sum at the input to the amplifier circuit (i.e., at the output of the sense element)
- Then, can compare the output of the sensed signal directly to the noise at this node to get the MDS

### Equivalent Input-Referred Voltage and Current Noise Sources
Equivalent Input $v$, $i$ Noise Generators

- Take a noisy 2-port network and represent it by a noiseless network with input $v$ and $i$ noise generators that generate the same total output noise.

![Equivalent Input $v$, $i$ Noise Generators](image)

- Remarks:
  2. $v_{eq}$ and $i_{eq}$ are generally correlated (since they are derived from the same sources).
  3. In many practical circuits, one of $v_{eq}$ and $i_{eq}$ dominates, which removes the need to address correlation.
  4. If correlation is important, it is easier to return to the original network with internal noise sources.

Calculation of $v_{eq}^2$ and $i_{eq}^2$

- To get $v_{eq}^2$ for a two-port:

  **Case I**
  - Short input, find $v_{0I}^2$ (or $i_{0I}^2$).
  - For eq. network, short input, find $v_{0II}^2$ (or $i_{0II}^2$).
  - Set $v_{0I}^2 = v_{0II}^2$ → solve for $v_{eq}^2$ (or $i_{0I}^2 = i_{0II}^2$).

- To get $i_{eq}^2$ for a two-port:

  **Case II**
  - Short input, find $i_{0I}^2$ (or $v_{0I}^2$).
  - For eq. network, short input, find $i_{0II}^2$ (or $v_{0II}^2$).
  - Set $i_{0I}^2 = i_{0II}^2$ → solve for $i_{eq}^2$ (or $v_{0I}^2 = v_{0II}^2$).
Calculation of $v_{eq}^2$ and $i_{eq}^2$ (cont)

b) To get $i_{eq}^2$ for a 2-port:

\[ \begin{align*}
\text{Noisy} & \quad v_{0I} \\
\text{Network} & \\
\text{Noiseless} & \quad i_{eq} \\
\end{align*} \]

1) Open input, find $v_{0I}^2$ (or $i_{0I}^2$)
2) Open input for eq. circuit, find $v_{0II}^2$ (or $i_{0II}^2$)
3) Set $v_{0I}^2 = v_{0II}^2 (i_{eq}^2)$ → solve for $i_{eq}^2$ (or $v_{0I}^2 = i_{0II}^2 (i_{eq}^2)$)

* Once the equivalent input-reflected noise generators are found, noise calculations become straightforward as long as the noise generators can be treated as uncorrelated.

Cases Where Correlation Is Not Important

* There are two common cases where correlation can be ignored:

1. Source resistance $R_S$ is small compared to input resistance $R_i$ i.e., voltage source input
2. Source resistance $R_S$ is large compared to input resistance $R_i$ i.e., current source input

1) $R_S$ = small (ideally = 0 for an ideal voltage source):

\[ \begin{align*}
R_S & \quad v_{eq}^2 \\
\text{Noiseless} & \quad i_{eq}^2 \\
\end{align*} \]

\[ \begin{align*}
\tilde{i}_{eq} & \quad \text{Current shorted out!} \\
\quad & \end{align*} \]

\[ \therefore \text{For } R_S \text{ small, } i_{eq}^2 \text{ can be neglected} \rightarrow \text{only } v_{eq}^2 \text{ is important!} \]

(Thus, we need not deal with correlation)
**Cases Where Correlation Is Not Important**

2) \( R_S = \text{large} \) (Ideally = \( \infty \) for an ideal current source)

\[
\begin{align*}
    \overline{v_{eq}^2} &= \overline{v_{eq}^2} \\
    \overline{i_{eq}^2} &= \overline{i_{eq}^2} \\
    \text{Voltage } \overline{v_{eq}^2} \text{ effectively "opened" out!} \\
    \overline{v_{eq}^2} &= \frac{R_{in}}{\infty + R_{in}} \overline{v_{eq}^2} = 0!
\end{align*}
\]

:. For \( R_S \) large, \( \overline{v_{eq}^2} \) can be neglected!

\[ \rightarrow \text{only } \overline{i_{eq}^2} \text{ is important!} \]

(... and again, we need not deal with correlation)

**Example: TransR Amplifier Noise**

\[
\begin{align*}
    \overline{v_{ia}^2} &= \overline{v_{ia}^2} \\
    \overline{i_{ia}^2} &= \overline{i_{ia}^2} \\
    \text{Case I:} \\
    \overline{v_{od}^2} &= \overline{v_{od}^2} \\
    \overline{\text{Input-referred current noise:}} \\
    \text{Open input; equals output voltage noise.} \\
    \text{Case I:} \\
    \overline{N_{ois1}} &= \lambda_{ia} R_f \\
    \overline{N_{ois2}} &= \lambda_{ia} R_f \\
    \overline{N_{ois3}} &= \lambda_{ia} R_f \\
    \text{Case II:} \\
    \overline{N_{ois1}^2} &= \overline{i_{ia}^2} R_f^2 \\
    \overline{N_{ois2}} &= \lambda_{ia} R_f \\
    \overline{N_{ois3}} &= \lambda_{ia} R_f \\
\end{align*}
\]
Example: TransR Amplifier Noise (cont)

Input-referred voltage noise:
Short inputs; equate output voltage noise

Case I:
\[
\frac{N_{0i}^2}{N_{ia}^2} \cdot \frac{V_{ia}^2}{R_f} \\
(\text{Both } i_{ia}^2 + i_f^2 \text{ are shorted out})
\]

Case II:
\[
\frac{N_{0i}^2}{N_{eq}^2} = \frac{V_{eq}^2}{R_f}
\]

Noiseless

To summarize, for a transresistance amplifier, the equivalent input-referred current and voltage noise generators are given by:

\[
i_{eq}^2 = i_{ia}^2 + i_f^2 + \frac{V_{ia}^2}{R_f} \\
V_{eq}^2 = V_{ia}^2
\]
Back to Gyro Noise & MDS

Example: Gyro MDS Calculation

\[ \vec{F}_c = m \vec{a}_c = m \cdot (2 \vec{x} \times \Omega) \]

- The gyro sense presents a large effective source impedance
- Currents are the important variable; voltages are "opened" out
- Must compare \( i_o \) with the total current noise \( i_{eqTOT} \) going into the amplifier circuit

\[ i_o \quad \text{v}_{eq}^2 \quad \frac{i_{eq}^2}{R_f} \]

\[ V_0 \]
Example: Gyro MDS Calculation (cont)

\[ \dot{\vec{x}} = m \ddot{\vec{x}} = m \cdot (2 \vec{x} \times \vec{\Omega}) \]

\[ F_c = m \ddot{\vec{x}} = m \cdot (2 \vec{x} \times \vec{\Omega}) \]

* First, find the rotation to \( i_o \) transfer function:

\[ \dot{x}_f = \frac{c_0}{k_0} \Theta_s(j \omega) F_c = \frac{2 \omega \chi_d \Theta_s(j \omega)}{\omega_c^2} \]

\[ \dot{x}_f = \frac{F_c}{2 \omega \chi_d \Theta_s(j \omega)} \]

\[ \dot{x}_f = \frac{2 \omega \chi_d \Theta_s(j \omega)}{\omega_c^2} \]

\[ \dot{x}_f = \frac{2 \omega \chi_d \Theta_s(j \omega)}{\omega_c^2} \]

Example: Gyro MDS Calculation (cont)

\[ i_0 = \frac{\omega}{\omega_c} \dot{x}_f = \frac{2 \omega \chi_d \Theta (j \omega)}{\omega_c} \]

When \( \Omega \ll \Omega_{\text{min}} \) is an MDS, \( i_0 = \dot{x}_f \) input-referred noise current entering the sense amplifier is \( \text{in pA/Hz}^{-1/2} \)

\[ \dot{x}_f = \frac{\omega}{\omega_c} \dot{x}_f \rightarrow \Omega_{\text{min}} = \frac{\dot{x}_f}{A} \left( \frac{3600 \text{ Hz}}{1 \text{ Hz}} \right) \left( \frac{1 \text{ pA}}{1 \text{ Hz}^{1/2}} \right) \]

\[ \text{Angle Random Walk} = \frac{1}{60} \Omega_{\text{min}} \left[ \frac{\text{pA} \cdot \text{Hz}^{1/2}}{\text{hr}} \right] \]

Easier to determine directional error as a function of elapsed time.
Example: Gyro MDS Calculation (cont)

Now, find the \( i_{eqTOT} \) entering the amplifier input:

\[
\begin{align*}
\dot{i}_{eqTOT} &= i_s + i_{eq} \\
\dot{i}_{eqTOT} &= \dot{i_s} + \frac{N_b}{R_x} + \frac{N_{ia}}{R_x} + \frac{N_f}{R_x} \\
\end{align*}
\]

\( \dot{f}_x = \frac{4kT}{R_x} \)

Brownian motion noise of the sense element determined entirely by the noise in \( R_x \).\( f_x \) easiest to convert to an all electrical current circuit.

Example: Gyro MDS Calculation (cont)

Thus:

\[
\frac{\dot{i}_{eqTOT}}{\Delta f} = \frac{4kT}{R_x} |\Theta| |\Omega| + \frac{4kT}{R_x} \frac{\dot{i}_{s}}{\Delta f} + \frac{\dot{i}_{eq}}{\Delta f} + \frac{N_{ia}}{R_x} \frac{1}{\Delta f}
\]

Learn to get these from EE247A, or just get them from a data sheet...
### Example ARW Calculation

**Example Design:**

Sensor Element:
- \( m = (100\, \mu m)(100\, \mu m)(20\, \mu m)(2300\, \text{kg/m}^3) = 4.6 \times 10^{-10}\, \text{kg} \)
- \( \omega_s = 2\pi(15\, \text{kHz}) \)
- \( \omega_d = 2\pi(10\, \text{kHz}) \)
- \( k_s = m \omega_s^2 = 4.09 \, \text{N/m} \)
- \( x_s = 20 \, \mu m \)
- \( Q_s = 50,000 \)
- \( V_p = 5\, \text{V} \)
- \( h = 20 \, \mu m \)
- \( d = 1 \, \mu m \)

Sensing Circuitry:
- \( R_s = 100\, \Omega \)
- \( i_{id} = 0.01 \, \text{pA/\sqrt{Hz}} \)
- \( V_{id} = 12 \, \text{nV/\sqrt{Hz}} \)

### LF356 Op Amp Data Sheet

**Features**
- Replace expensive hybrid and module FET op amps
- Rugged JFETs allow blow-out free handling compared with MOSFET input devices
- Excellent for low noise applications using either high or low source impedance—very low 1/f corner
- Offset adjust does not degrade drift or common-mode rejection as in most monolithic amplifiers
- New output stage allows use of large capacitive loads (5,000 pF) without stability problems
- Internal compensation and large differential input voltage capability

**Applications**
- Precision high speed integrators
- Fast D/A and A/D converters
- High impedance buffers
- Wideband, low noise, low drift amplifiers

**Uncommon Features**

<table>
<thead>
<tr>
<th>LF355</th>
<th>LF356</th>
<th>LF357</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniax</td>
<td>Ultra</td>
<td>Audio</td>
</tr>
<tr>
<td>LF155</td>
<td>LF156</td>
<td>LF257</td>
</tr>
<tr>
<td>LF256</td>
<td>LF356</td>
<td>LF357</td>
</tr>
</tbody>
</table>

- Extremely fast settling time to 0.01%:
  - 4
  - 1.5
  - 1.5

- Fast slew:
  - 5
  - 12
  - 50

- Wide gain bandwidth:
  - 2.5
  - 5
  - 20

- Low input noise voltage:
  - 20
  - 12
  - 12

\[ \frac{dV_{in}}{df} = 12 \text{nV/Hz} \]

\[ \frac{dV_{out}}{df} \approx 0.01 \text{pA/Hz} \]
Example ARW Calculation (cont)

Get relation rate to output current scale factor:

\[ A = \frac{2 \pi d Q_{x} Q_{y} \eta}{\omega_{s}} \left| \Theta(j \omega) \right| = \frac{2 \pi d (\omega_{s}/\omega_{t})}{\omega_{s}} + \frac{j \pi}{\omega_{s}} + \frac{j \pi}{\omega_{s}} \]

\[ \Theta(j \omega) = \frac{\omega_{s} / \omega_{t} + j \pi}{\omega_{s} / \omega_{t} + j \pi} = \frac{j \pi (\omega_{s} / \omega_{t})}{(\omega_{s} / \omega_{t})^{2} + (j \pi)^{2}} = \frac{j \pi}{1.25 \times 10^{8} + j \pi} \]

\[ \Theta(j \omega) = \frac{3j \pi}{(1.25 \times 10^{8})^{2} + (j \pi)^{2}} = 0.000024 \]

Then, get noise:

\[ \frac{\frac{\Delta V_{n}}{\Delta f}}{V_{c}} = \frac{1 \times kT}{R_{0}} \left| \Theta(j \omega) \right|^{2} + \frac{1 \times kT}{R_{0} + j \pi} \frac{H_{n}}{\Delta f} + \frac{\frac{1}{2} \pi}{SF_{c}} \frac{1}{\Delta f} \]

Example ARW Calculation (cont)

\[ R_{x} = \frac{2 \pi \omega_{0} m}{Q_{m} h_{e}} = \frac{2 \pi (5000 \pi / 0.45 \times 10^{-9})}{(5000 \pi / 2.83 \times 10^{-12})} = 110.6 k \Omega \]

\[ \frac{\frac{\Delta V_{n}}{\Delta f}}{V_{c}} = \frac{1 \times 2 \times 10^{-12}}{(110.6 \Omega)} \frac{(0.45 \times 10^{-9})^{2}}{(110.6 \Omega)} + \frac{1 \times 10^{-10}}{(110.6 \Omega)}(0.83 \pi)^{2} + \frac{1 \times 10^{-10}}{(110.6 \Omega)}(0.83 \pi)^{2} \]

\[ 8.6 \times 10^{-14} A^{2} / Hz \]

\[ \text{Senser element noise insignificant} \]

\[ \frac{\frac{\Delta V_{n}}{\Delta f}}{V_{c}} = 1.6 \times 10^{-26} A^{2} / Hz \rightarrow \frac{\frac{\Delta V_{n}}{\Delta f}}{V_{c}} = 1.3 \times 10^{-12} A / \sqrt{Hz} \]

And finally:

\[ \text{ARW} = \frac{1}{60} \left( \frac{9478}{60} \right) = \frac{1}{60} \left( 157 \% / hr \right) = \text{Almost thermal around in 1 hour!} \]
What if $\omega_d = \omega_s$?

If $\omega_d = \omega_s = 15000$, then $|G(j\omega_d)|^2 = 1$ and

$$A = 2 \frac{\omega_d}{\omega_s} \frac{Q_f}{Q_e} \frac{\eta_e}{\eta_f} |G(j\omega_d)|^2 + 2 Q_f \frac{X_d \eta_e}{\eta_f} = 2 (50) (20 \mu) (2) (10000) = 1.77 \times 10^{-7} C$$

$$\frac{\Delta \omega_{eq}}{\Delta f} = \frac{\frac{1.64 \times 10^{24}}{(110.6)} \left(1 + \frac{1.66 \times 10^{10}}{1M} + \frac{0.01}{1M} \right) \left(\frac{22}{140} \right) \left(\frac{121}{1M} \right) \left(\frac{1.5 \times 10^{-25} A^2/Hz}{1.66 \times 10^{-26} A^2/Hz} \right) \left(\frac{1 \times 10^{-25} A^2/Hz}{1.44 \times 10^{-28} A^2/Hz} \right)}{1.67 \times 10^{-2} \frac{5}{2} A^2/Hz} \rightarrow \Delta \omega_{eq} \Delta f = 4.08 \times 10^{-12} A/HHz$$

$$\Delta \omega_{min} = \frac{\Delta \omega_{eq}}{A} \left(\frac{360 \pi}{hr} \right) \left(\frac{180}{\pi} \right) = \frac{4.08 \times 10^{-12}}{1.77 \times 10^{-7}} \left(360 \pi \right) \left(144 \pi \right) = 0.476 \% / hr$$

And finally:

$$AR = \frac{1}{60} \Delta \omega_{min} = \frac{1}{60} (0.476) = 0.0079 \% / hr = ARW$$

$$\Rightarrow \text{Navigation grade!}$$