EE249 Discussion

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What are Petri Nets

- A graphical & modeling tool.
- Describe systems that are concurrent, asynchronous, distributed, parallel, nondeterministic, and/or stochastic.
- Graphically as a visual-communication aid similar to flow charts, block diagrams, and networks.
- Mathematically for state equations, algebraic equations, and behavioral models.
Application Areas

- Performance evaluation
- Communication protocols
- Modeling and analysis of distributed-software systems
- Distributed database systems
- Concurrent and parallel programs
- ...
- Note: special modifications or restrictions suited to the particular application are often necessary.
Formal Definition

A Petri net is a 5-tuple, $PN = (P, T, F, W, M_0)$ where:

- $P = \{p_1, p_2, \cdots, p_m\}$ is a finite set of places,
- $T = \{t_1, t_2, \cdots, t_n\}$ is a finite set of transitions,
- $F \subseteq (P \times T) \cup (T \times P)$ is a set of arcs (flow relation),
- $W: F \rightarrow \{1, 2, 3, \cdots\}$ is a weight function,
- $M_0: P \rightarrow \{0, 1, 2, 3, \cdots\}$ is the initial marking,
- $P \cap T = \emptyset$ and $P \cup T \neq \emptyset$.

A Petri net structure $N = (P, T, F, W)$ without any specific initial marking is denoted by $N$.

A Petri net with the given initial marking is denoted by $(N, M_0)$. 
Transition Rule

- A transition $t$ is enabled if each input place $p$ of $t$ is marked with enough tokens.
- A enabled transition may or may not fire.
- A firing of an enabled transition $t$ removes tokens from each input place and adds tokens to each output place.
An Example of Firing Rule

Fig. 1. Example 1: An illustration of a transition (firing) rule: (a) The marking before firing the enabled transition $t$. (b) The marking after firing $t$, where $t$ is disabled.
Strict Transition Rule

- K(p): maximum number of tokens that place p can hold at any time.
- The number of tokens in each output place p of t cannot exceed its capacity K(p) after firing t.
- Weak transition rule: Without the above capacity constraint.
An Example

Fig. 2. Example 2: An illustration of the complementary-place transformation: (a) A finite-capacity net \((N, M_0)\). (b) The net \((N', M'_0)\) after the transformation. (c) The reachability graph for the net \((N, M_0)\) shown in (a).
An Example (Cont’d)

All properties associated with a finite-capacity net can be discussed in terms of those with an infinite-capacity net using the complementary transformation.
Deterministic Parallel Activities

Fig. 6. A Petri net (a marked graph) representing deterministic parallel activities.

V.S.

Fig. 5. A Petri-net structure called a conflict, choice, or decision. It is a structure exhibiting nondeterminism.
Reachability

- A marking Mn is said reachable form a marking M0 if there exists a sequence of firings that transforms M0 to Mn.
- R(N,M0) or R(M0): the set of all possible markings reachable from M0.
- L(N,M0) or L(M0): the set of all possible firing sequences from M0.
- Reachability problem: finding if Mn∈R(M0)
Boundedness

- **k-Bounded**: if the number of tokens in each place does not exceed a finite number $k$ for any marking reachable from $M_0$, $M(p) \leq k$, for every place $p$ and every marking $M \in R(M_0)$.

- **Safe**: if 1-bounded

- Guaranteed no overflow in buffers and registers.
Liveness

- Live: if it is possible to fire any transition of the net by progressing through some further firing sequence.
- Guarantees deadlock-free operation.
Reversibility and Home State

- Reversible: if M₀ is reachable from M. One can always get back to M₀.
- M’ is said a home state if M’ is reachable from M.
Coverability

- A marking $M$ is said coverable if there exists a marking $M'$ in $R(M0)$ such that $M'(p) \geq M(p)$ for each $p$ in the net.
Persistence

- Persistent: if, for any two enabled transitions, the firing of one transition will not disable the other.
Synchronic Distance

- A metric closely related to a degree of mutual dependence between two events.

- Definition of synchronic distance between two transitions $t_1$ and $t_2$:

\[
d_{12} = \max_{\sigma} |\bar{\sigma}(t_1) - \bar{\sigma}(t_2)|
\]  

(1)

where $\sigma$ is a firing sequence starting at any marking $M$ in $R(M_0)$ and $\bar{\sigma}(t_i)$ is the number of times that transition $t_i$, $i = 1, 2$ fires in $\sigma$. 


Fairness

- Bounded-fair (B-fair) of two transitions $t_1$ and $t_2$: if the maximum number of times that either one can fire while the other is not firing is bounded.

- Unconditionally fair at a firing sequence $\sigma$: if the sequence is finite or every transition in the net appears infinitely often in $\sigma$. 
Coverability Tree

ω: infinity

Fig. 18. (a) The coverability tree of the net shown in Fig. 16. (b) The coverability graph of the net shown in Fig. 16.
Properties by Coverability Tree

- Bounded and finite iff $\omega$ does not appear in any node labels.
- Safe iff only 0’s and 1’s appear in node labels.
- A transition $t$ is dead iff it does not appear as an arc label.
- If $M$ is reachable from $M_0$, then there exists a node labeled $M'$ such that $M \leq M'$. 

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Incidence Matrix

- For a Petri net $N$ with $n$ transitions and $m$ places, the incidence matrix $A = [a_{ij}]$ is an $n \times m$ matrix of integers.
Reduction Rules for Analysis

Fig. 22. Six transformations preserving liveness, safeness, and boundedness.
Subclasses of Petri Nets

- State machine (SM)
- Marked graph (MG)
- Free-choice (FC)
- Extended free-choice net (EFC)
- Asymmetric choice net (AC)
An Example

(a) SM, MG

(b) MG, SM

(c) $p_1$, $p_2$

(d) $p_1$, AC, EFC

(e) $p_1$, $p_2$, PN, AC
Fig. 25. Key structures characterizing subclasses of Petri nets and their Venn diagram, where MG, SM, etc., denote nonMG, nonSM, etc.
Marked Directed Graph (G,M0)

- Arcs correspond to places, nodes to transitions, and tokens are placed on arcs.

Fig. 28. The marked graph representation of a communication protocol shown in Fig. 9 and used for Example 14.
Siphon and Trap

\[ S = \{ t_1 \} \]
\[ S^* = \{ t_1, t_2 \} \]
\[ S \subseteq S^* \]

\[ Q^* = \{ t_1 \} \]
\[ Q = \{ t_1, t_2 \} \]
\[ Q^* \subseteq Q \]

(a)

(b)

Fig. 30. Illustration of (a) a siphon and (b) a trap.
Expansion Rules

Fig. 45. Illustration of the five expansion rules (a) series expansion, (b) parallel expansion, (c) unique circuit expansion, (d) V-Y expansion, and (e) separate graph expansion

Fig. 46. Illustration of the use of expansion rules for LSMG synthesis.
Synthesis of Synchronic Distance Matrix

\[ D = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 0 & 1 & 3 & 6 & 6 & 5 & 6 \\ 2 & 1 & 0 & 2 & 5 & 5 & 4 & 5 \\ 3 & 3 & 2 & 0 & 3 & 3 & 2 & 3 \\ 6 & 5 & 4 & 2 & 5 & 1 & 0 & 1 \\ 7 & 6 & 5 & 3 & 6 & 2 & 1 & 0 \end{bmatrix} \]

\[ d_{\text{max}} \]

for \( i_0 = 1, 4, 7 \)

\[
\begin{array}{|c|c|c|}
\hline
i_0 & j_r & d_{\text{min}} \\
\hline
1 & 2 & 1 \\
4 & 3 & 3 \\
5 & 6 & 1 \\
7 & 6 & 1 \\
\hline
\end{array}
\]

Fig. 50. Example 17: Synthesis of a synchronic distance matrix.
An Example of Petri Nets Extensions: High-Level Nets

Fig. 57. Illustration of transition firing rule in a high-level net: (a) before firing, (b) after firing with substitution \( \{a|x, b|y, c|z\} \), and (c) after firing with substitution \( \{a|x, a|y, b|z\} \).

Fig. 58. The unfoleded net of the high-level net shown in Fig. 57.