EE249 Discussion
Petri Nets: Properties, Analysis and Applications - T. Murata

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What are Petri Nets

- A graphical & modeling tool.
- Describe systems that are concurrent, asynchronous, distributed, parallel, nondeterministic, and/or stochastic.
- Graphically as a visual-communication aid similar to flow charts, block diagrams, and networks.
- Mathematically for state equations, algebraic equations, and behavioral models.

Application Areas

- Performance evaluation
- Communication protocols
- Modeling and analysis of distributed-software systems
- Distributed database systems
- Concurrent and parallel programs
- …
- Note: special modifications or restrictions suited to the particular application are often necessary.

Formal Definition

A Petri net is a 5-tuple, $PN = (P, T, F, W, M_0)$ where:

$P = \{p_1, p_2, \cdots, p_m\}$ is a finite set of places,
$T = \{t_1, t_2, \cdots, t_n\}$ is a finite set of transitions,
$F \subseteq (P \times T) \cup (T \times P)$ is a set of arcs (flow relation),
$W: F \rightarrow \{1, 2, 3, \cdots\}$ is a weight function,
$M_0: P \rightarrow \{0, 1, 2, 3, \cdots\}$ is the initial marking,
$P \cap T = \varnothing$ and $P \cup T \neq \varnothing$.

A Petri net structure $N = (P, T, F, W)$ without any specific initial marking is denoted by N.

A Petri net with the given initial marking is denoted by (N, M_0).

Transition Rule

- A transition t is enabled if each input place p of t is marked with enough tokens.
- A enabled transition may or may not fire.
- A firing of an enabled transition t removes tokens from each input place and adds tokens to each output place.

An Example of Firing Rule

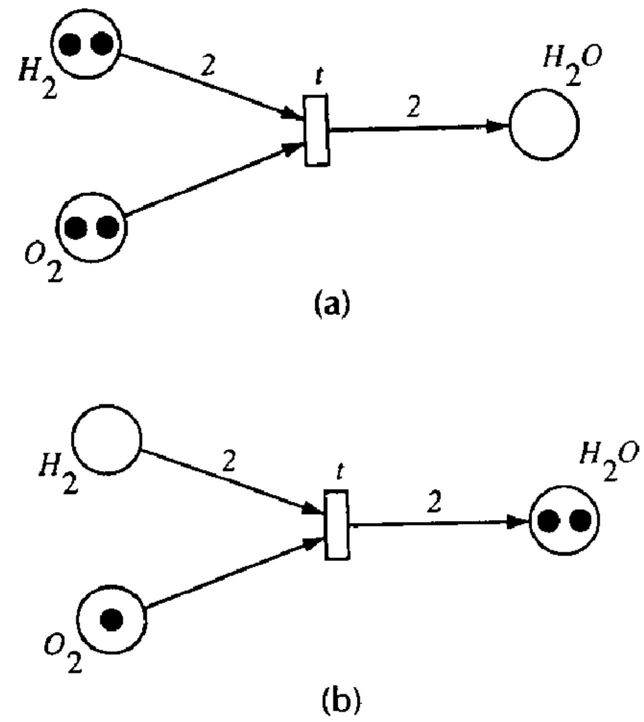
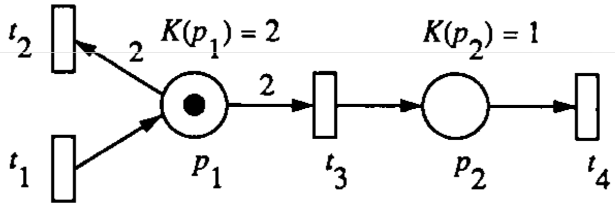


Fig. 1. Example 1: An illustration of a transition (firing) rule: (a) The marking before firing the enabled transition *t*. (b) The marking after firing *t*, where *t* is disabled.

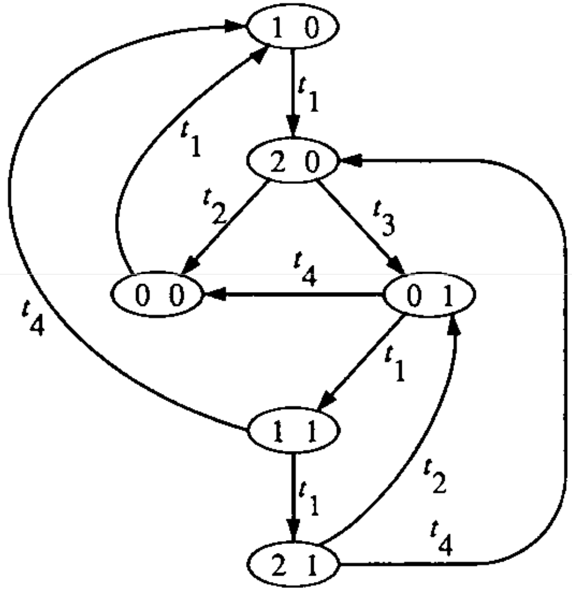
Strict Transition Rule

- K(p): maximum number of tokens that place p can hold at any time.
- The number of tokens in each output place p of t cannot exceed its capacity K(p) after firing t.
- Weak transition rule: Without the above capacity constraint.

An Example



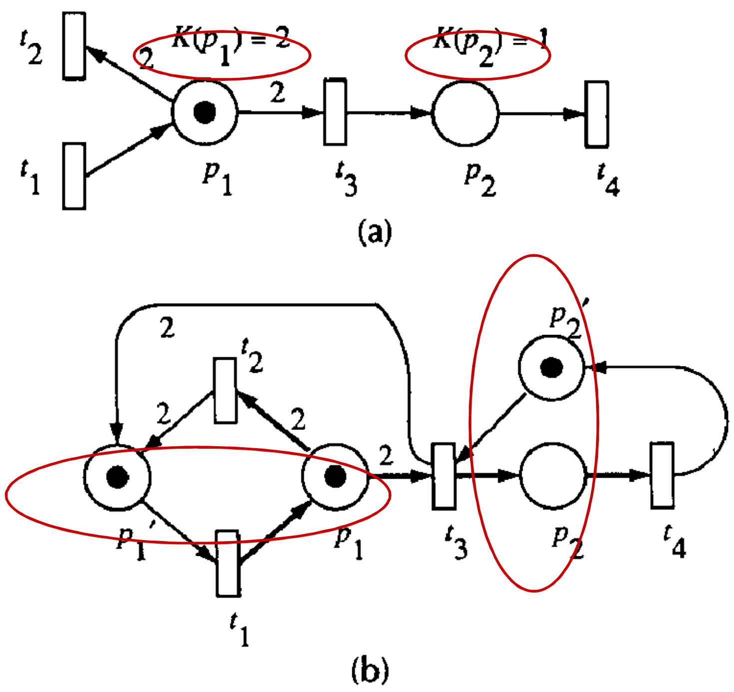
(a)



(c)

Fig. 2. Example 2: An illustration of the complementary-place transformation: (a) A finite-capacity net (N, M_0). (b) The net (N', M'_0) after the transformation. (c) The reachability graph for the net (N, M_0) shown in (a).

An Example (Cont'd)



All properties associated with a finite-capacity net can be Discussed in terms of those with an infinite-capacity net Using the complementary transformation.

Deterministic Parallel Activities

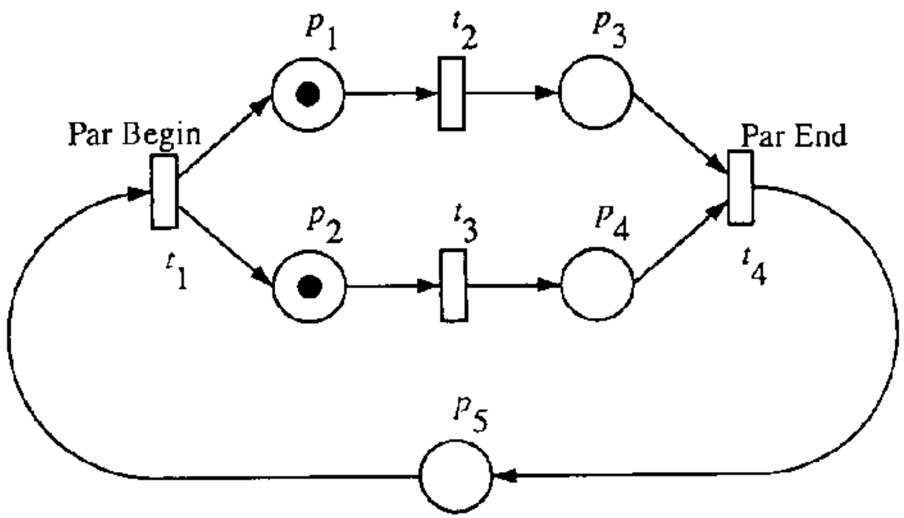


Fig. 6. A Petri net (a marked graph) representing deterministic parallel activities.

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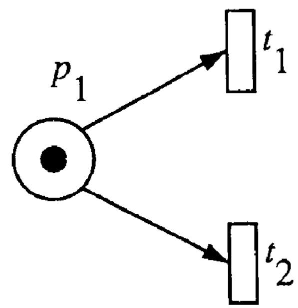


Fig. 5. A Petri-net structure called a conflict, choice, or decision. It is a structure exhibiting nondeterminism.

Reachability

- A marking Mn is said reachable form a marking M0 if there exists a sequence of firings that transforms M0 to Mn.
- R(N,M0) or R(M0): the set of all possible markings reachable from M0.
- L(N,M0) or L(M0): the set of all possible firing sequences from M0.
- Reachability problem: finding if $Mn \in R(M0)$

Boundedness

- k-Bounded: if the number of tokens in each place does not exceed a finite number k for any marking reachable form M0, M(p)≤k, for every place p and every marking M∈R(M0).
- Safe: if 1-bounded
- Guaranteed no overflow in buffers and registers.

Liveness

- Live: if it is possible to fire any transition of the net by progressing through some further firing sequence.
- Guarantees deadlock-free operation.

Reversibility and Home State

- Reversible: if M0 is reachable from M. One can always get back to M0.
- M' is said a home state if M' is reachable from M.

Coverability

- A marking M is said coverable if there exists a marking M’ in R(M0) shuch that M’(p)≥M(p) for each p in the net.

Persistence

- Persistent: if, for any two enabled transitions, the firing of one transition will not disable the other.

Synchronic Distance

- A metric closely related to a degree of mutual dependence between two events.
- Definition of synchronic distance between two transitions t1 and t2:

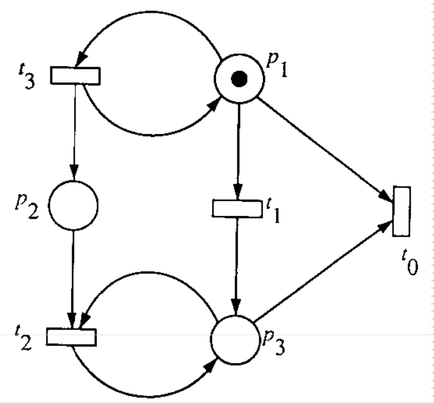
$$d_{12} = \max_{\sigma} \left| \overline{\sigma}(t_1) - \overline{\sigma}(t_2) \right| \qquad (1)$$

where σ is a firing sequence starting at any marking M in $R(M_0)$ and $\overline{\sigma}(t_i)$ is the number of times that transition t_i, $i = 1, 2$ fires in σ.

Fairness

- Bounded-fair (B-fair) of two transitions t1 and t2: if the maximum number of times that either one can fire while the other is not firing is bounded.
- Unconditionally fair at a firing sequence σ: if the sequence is finite or every transition in the net appears infinitely often in σ.

Coverability Tree



ω: infinity

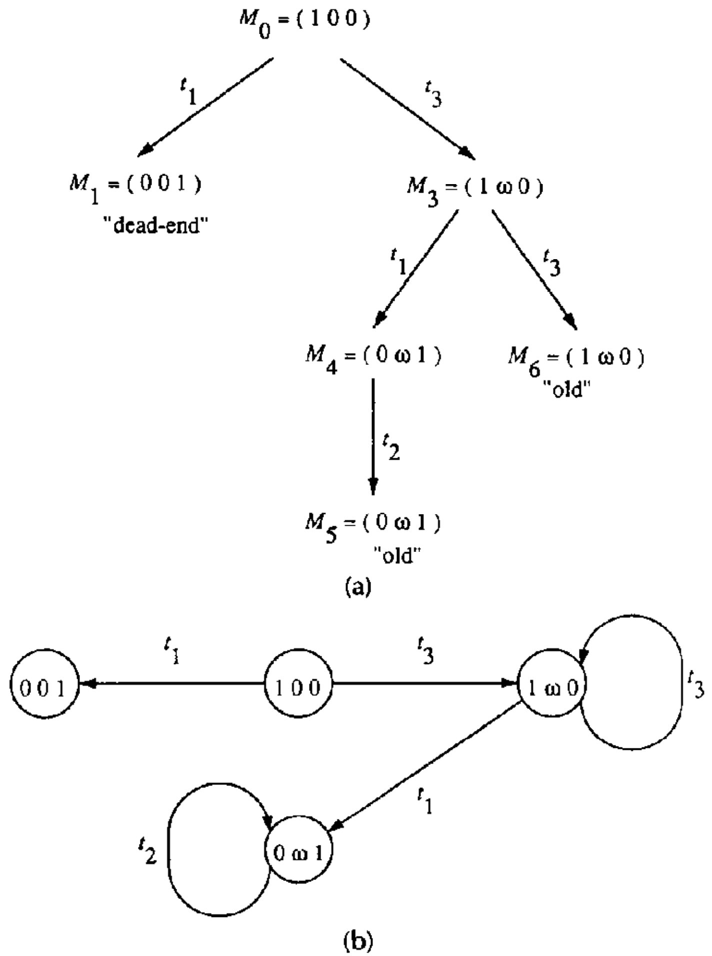


Fig. 18. (a) The coverability tree of the net shown in Fig. 16. (b) The coverability graph of the net shown in Fig. 16.

Properties by Coverability Tree

- Bounded and finite iff ω does not appear in any node labels.
- Safe iff only 0's and 1's appear in node lables.
- A transition t is dead iff it does not appear as an arc label.
- If M is reachable from M0, then there exists a node labeled M' such that M≤ M'.

Incidence Matrix

- For a Petri net N with n transitions and m places, the incidence matrix A=[aij] is an n x m matrix of integers.

Reduction Rules for Analysis

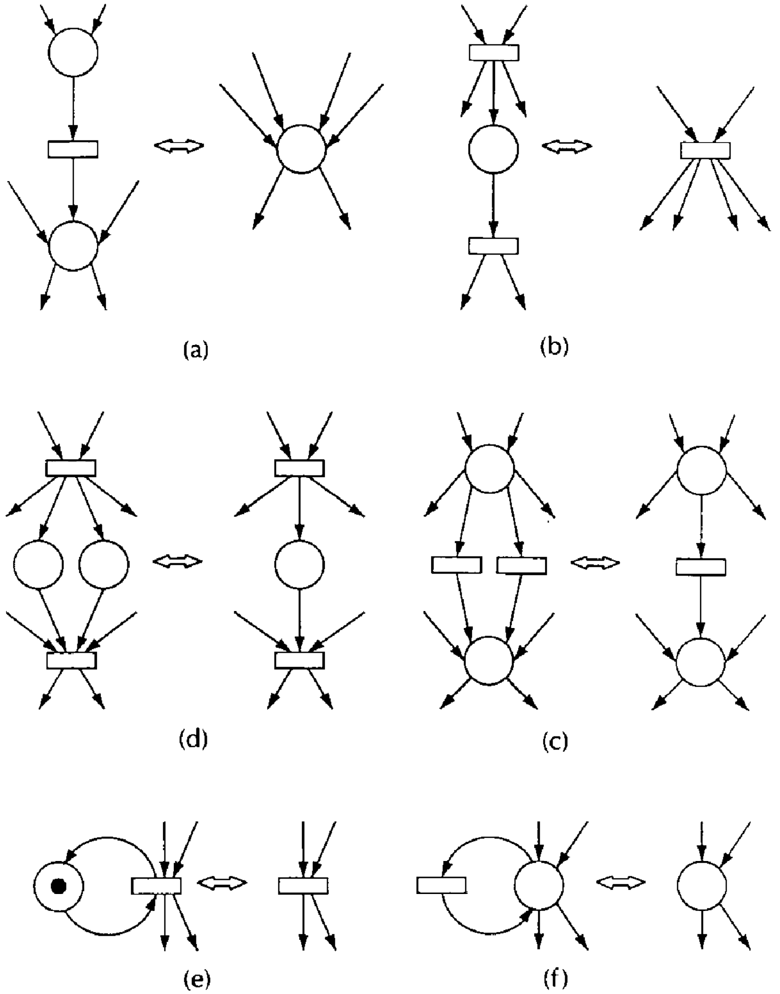
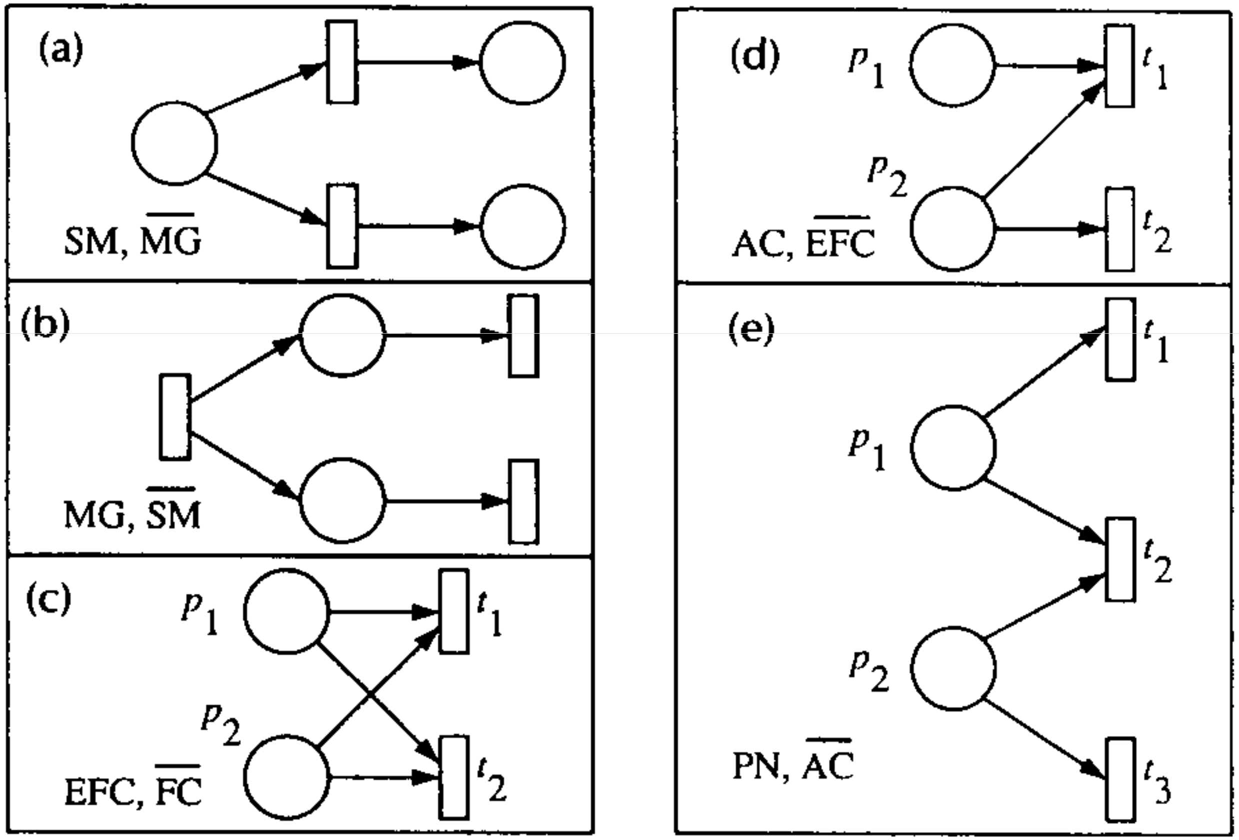


Fig. 22. Six transformations preserving liveness, safeness, and boundedness.

Subclasses of Petri Nets

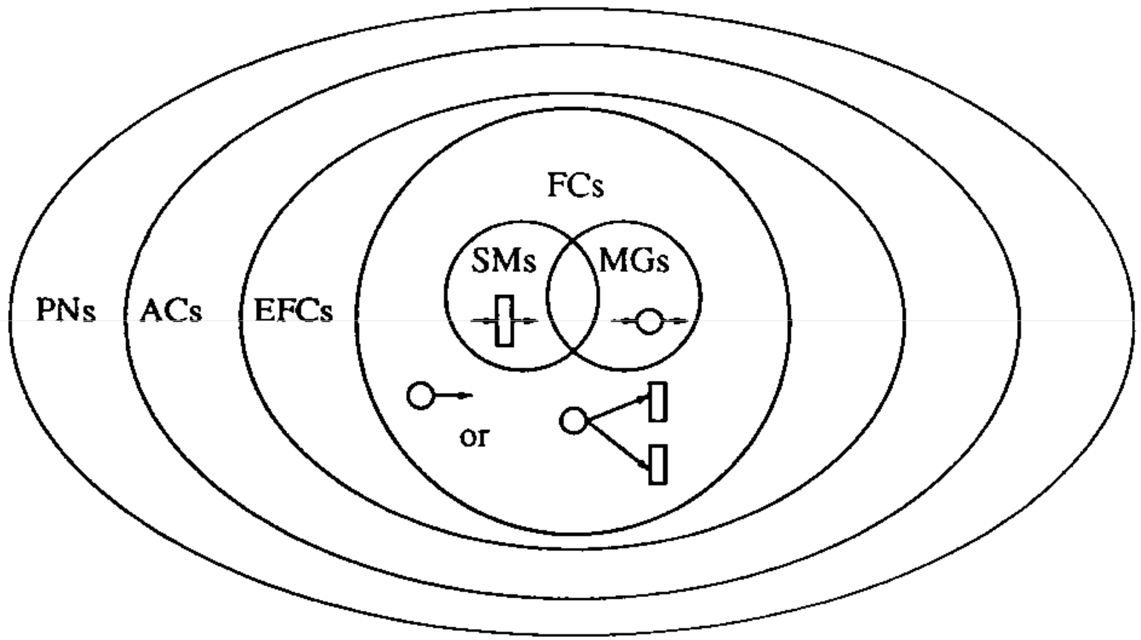
- State machine (SM)
- Marked graph (MG)
- Free-choice (FC)
- Extended free-choice net (EFC)
- Asymmetric choice net (AC)

An Example



(a)
SM, $\overline{MG}$
(b)
MG, $\overline{SM}$
(c)
p_1
t_1
p_2
t_2
EFC, $\overline{FC}$
(d)
p_1
t_1
p_2
t_2
AC, $\overline{EFC}$
(e)
t_1
p_1
t_2
p_2
PN, $\overline{AC}$
t_3

Venn Diagram



(f)

Fig. 25. Key structures characterizing subclasses of Petri nets and their Venn diagram, where $\overline{MG}$, $\overline{SM}$, etc., denote nonMG, nonSM, etc.

Marked Directed Graph (G,M0)

- Arcs correspond to places, nodes to transitions, and tokens are placed on arcs.

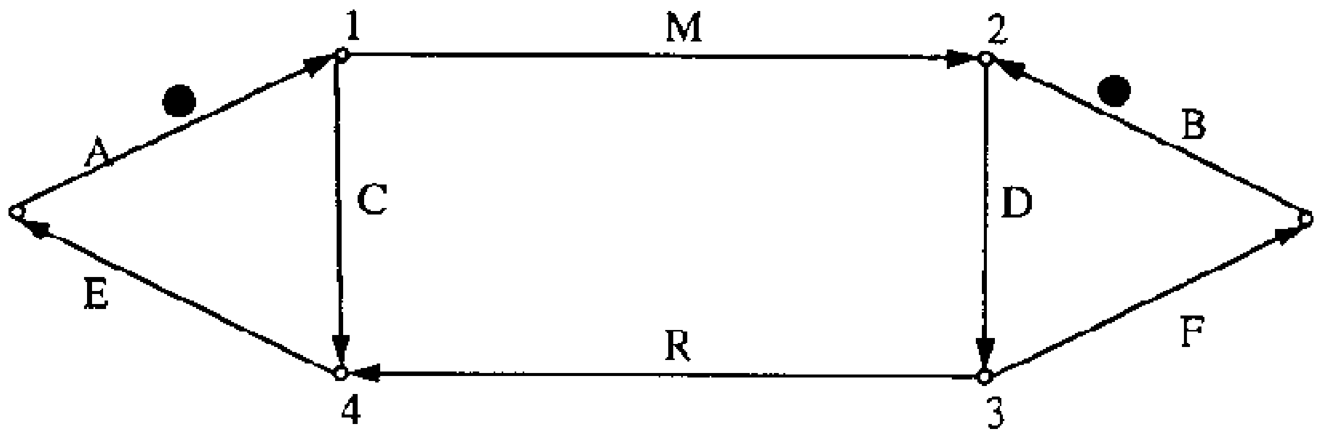
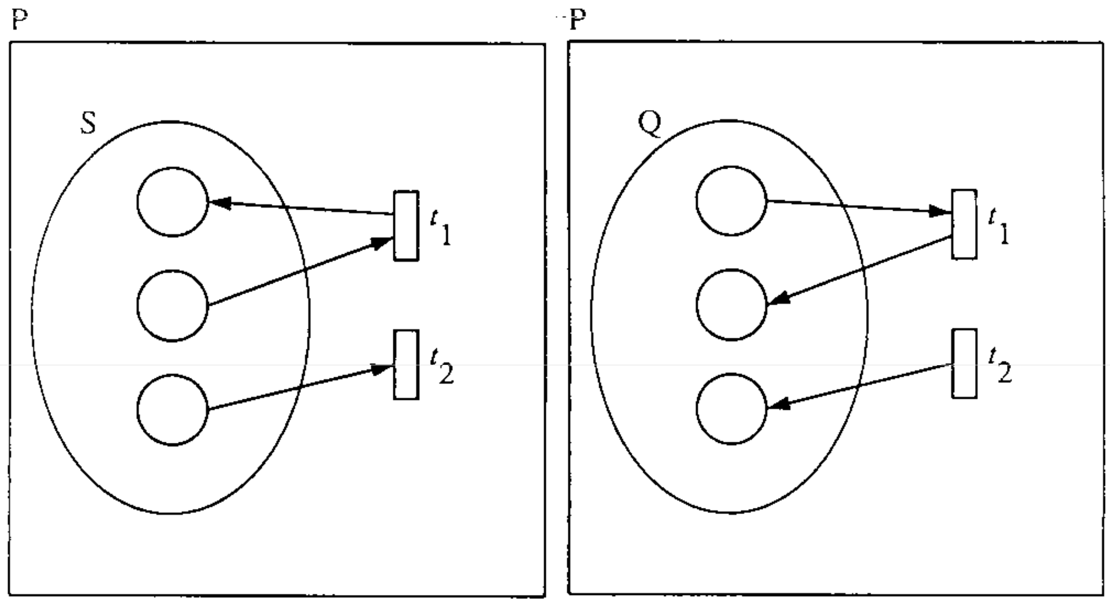


Fig. 28. The marked graph representation of a communication protocol shown in Fig. 9 and used for Example 14.

Siphon and Trap



$\bullet S = \{ t_1 \}$

$S\bullet = \{ t_1, t_2 \}$

$\bullet S \subseteq S\bullet$

(a)

$Q\bullet = \{ t_1 \}$

$\bullet Q = \{ t_1, t_2 \}$

$Q\bullet \subseteq \bullet Q$

(b)

Fig. 30. Illustration of (a) a siphon and (b) a trap.

Expansion Rules

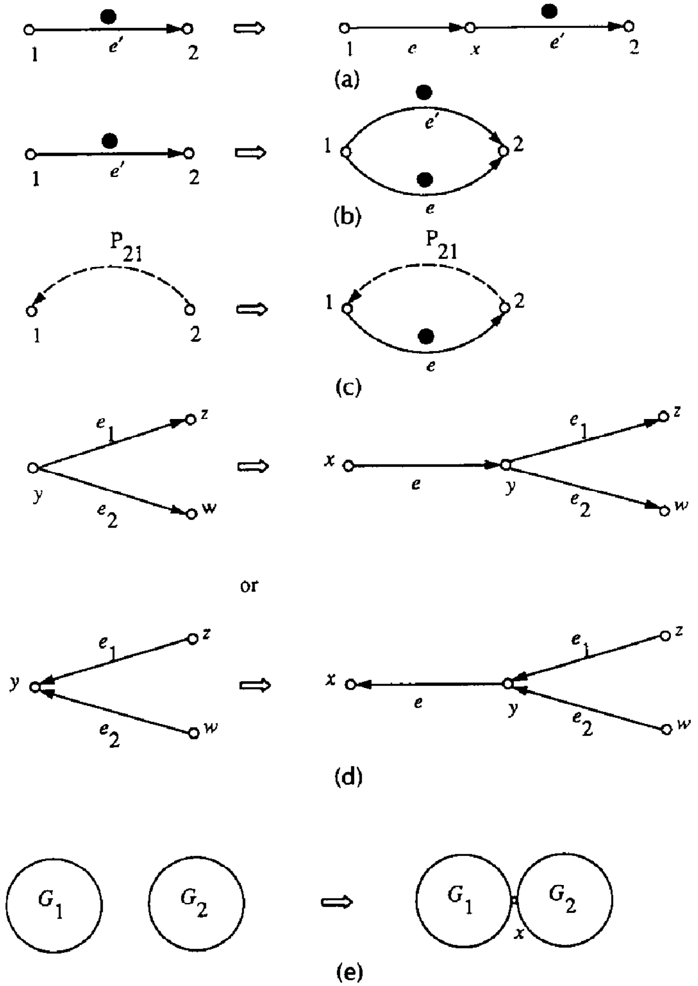


Fig. 45. Illustration of the five expansion rules (a) series expansion, (b) parallel expansion, (c) unique circuit expansion, (d) V–Y expansion, and (e) separate graph expansion

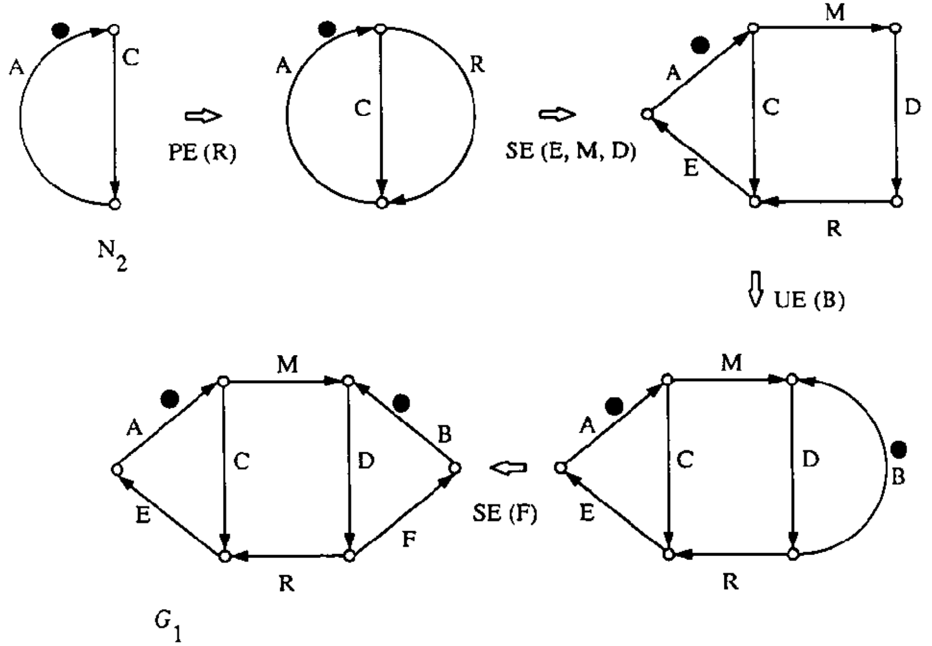
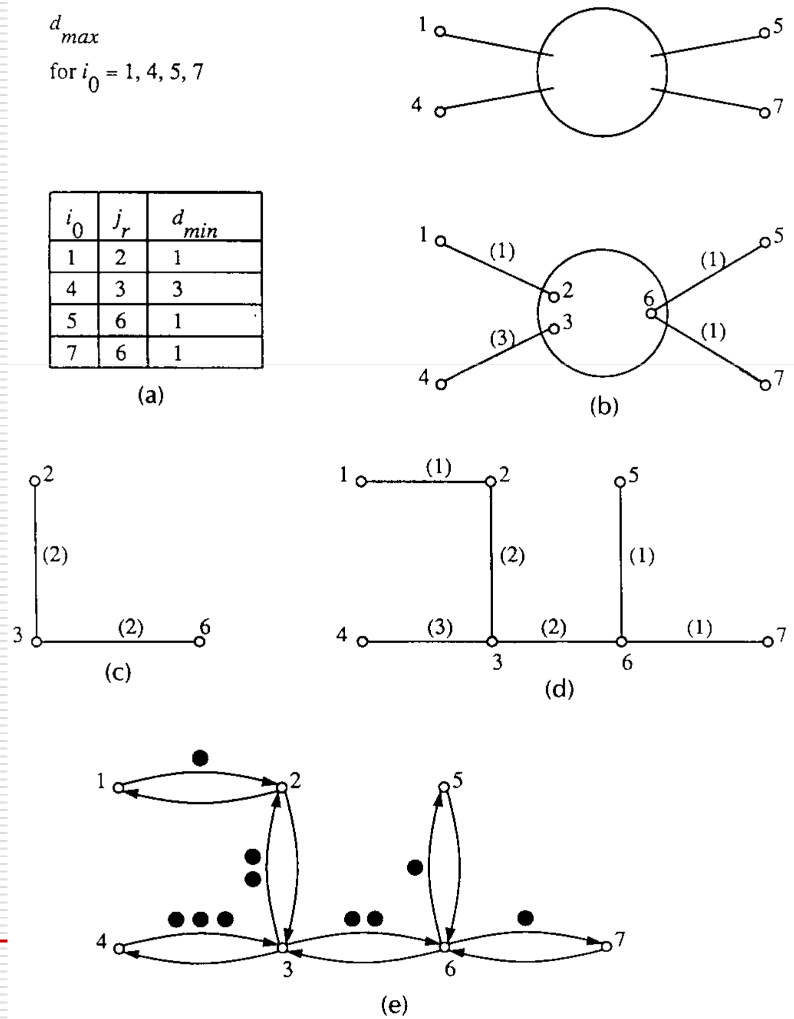


Fig. 46. Illustration of the use of expansion rules for LSMG synthesis.

Synthesis of Synchronic Distance Matrix

$$
\begin{array}{c} \\ \\ \\ \\ D = \end{array}
\begin{array}{c|ccccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
1 & 0 & 1 & 3 & 6 & 6 & 5 & 6 \\
2 & 1 & 0 & 2 & 5 & 5 & 4 & 5 \\
3 & 3 & 2 & 0 & 3 & 3 & 2 & 3 \\
4 & 6 & 5 & 3 & 0 & 6 & 5 & 6 \\
5 & 6 & 5 & 3 & 6 & 0 & 1 & 2 \\
6 & 5 & 4 & 2 & 5 & 1 & 0 & 1 \\
7 & 6 & 5 & 3 & 6 & 2 & 1 & 0
\end{array}.
$$

$$
D_1 = \begin{array}{c|ccc|}
 & 2 & 3 & 6 \\
\hline
2 & 0 & 2 & 4 \\
3 & 2 & 0 & 2 \\
6 & 4 & 2 & 0
\end{array}
$$



i_0	j_r	d_{min}
1	2	1
4	3	3
5	6	1
7	6	1

Fig. 50. Example 17: Synthesis of a synchronic distance matrix.

An Example of Petri Nets Extensions: High-Level Nets

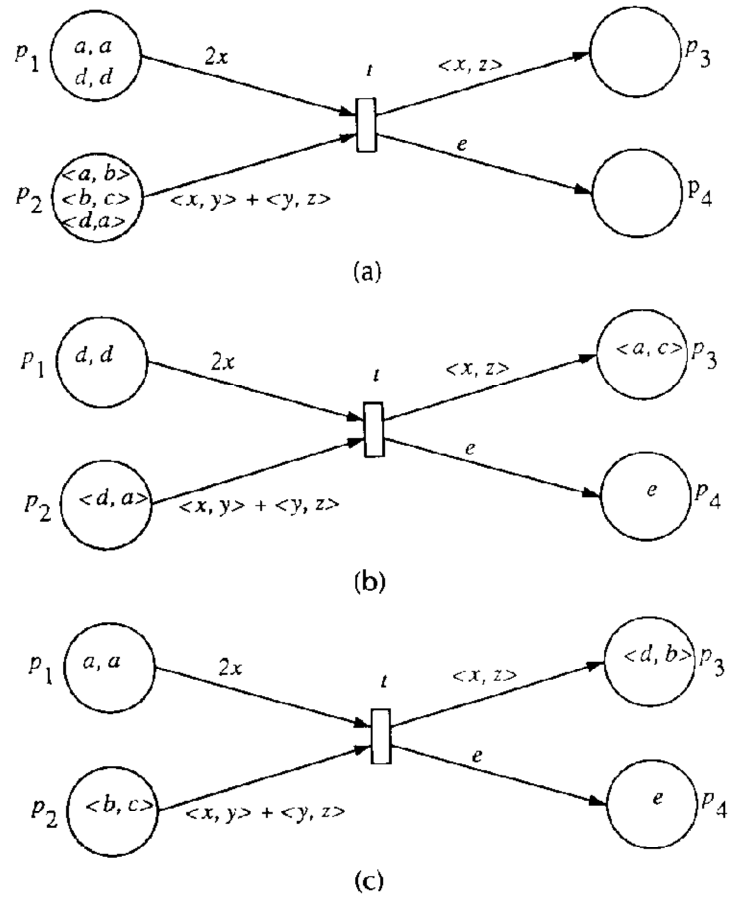


Fig. 57. Illustration of transition firing rule in a high-level net: (a) before firing, (b) after firing with substitution $\{a|x, b|y, c|z\}$, and (c) after firing with substitution $\{d|x, a|y, b|z\}$.

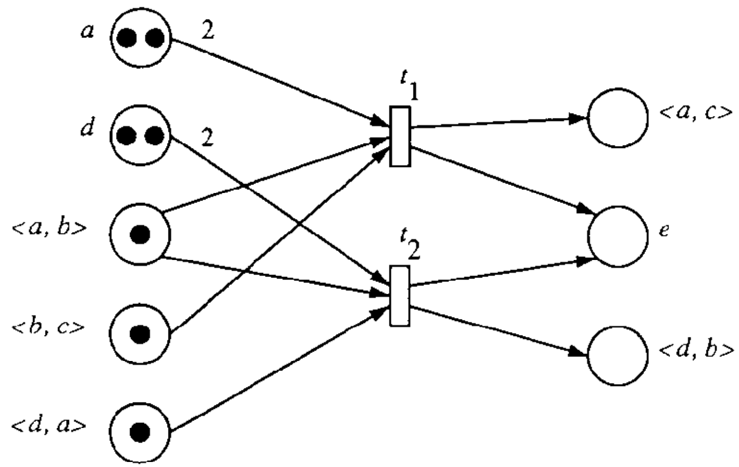


Fig. 58. The unfoleded net of the high-level net shown in Fig. 57.