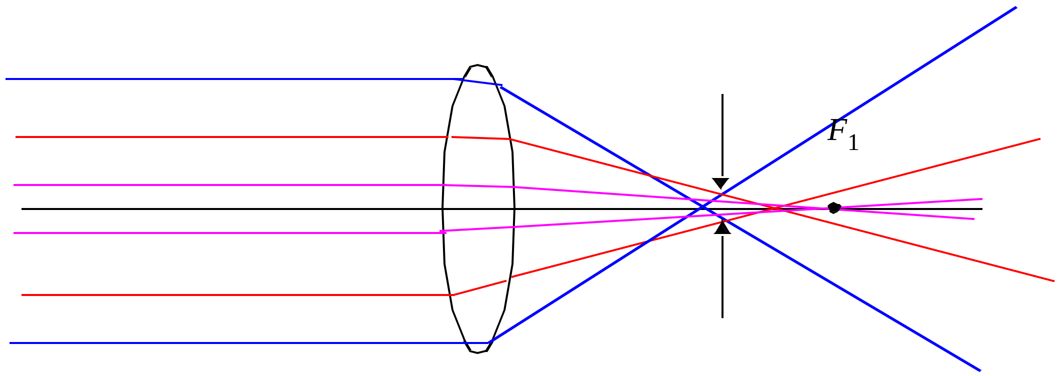
Chapter 6
Aberrations

As we have seen, spherical lenses only obey Gaussian lens law in the paraxial approximation. Deviations from this ideal are called aberrations.

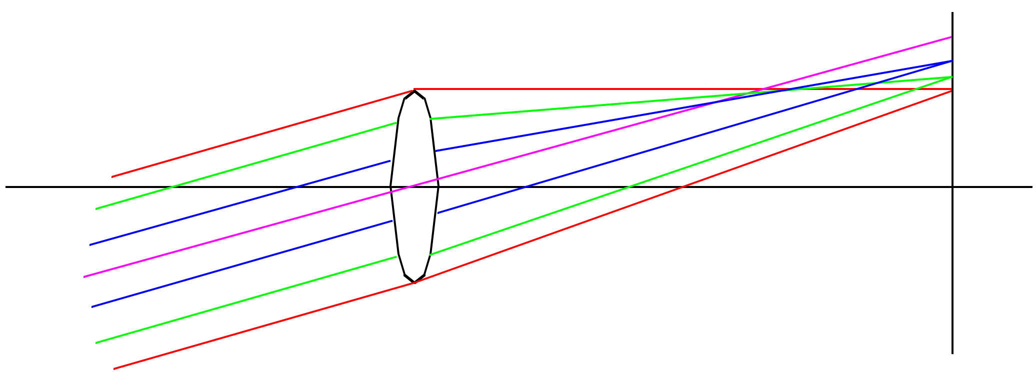


Rays toward the edge of the pupil (even parallel to the axis) violate the paraxial condition on the incidence angle at the first surface. They focus closer (for a biconvex lens) than F_1. No truly sharp focus occurs. The least blurred spot (smallest disc) is called circle of least confusion, or best focus. This form of symmetric aberration is spherical aberration.

There are many forms of aberration.

Coma: Variation of magnification with aperture.

Rays passing through edge portions of the pupil are imaged at a different height than those passing through the center.



Map of Rays in Pupil

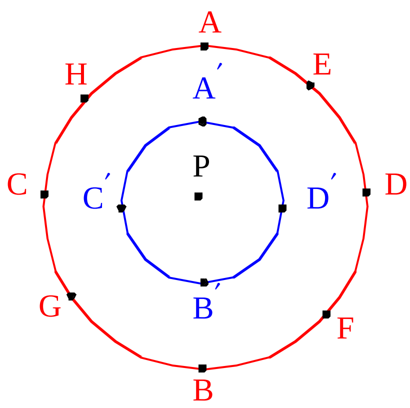
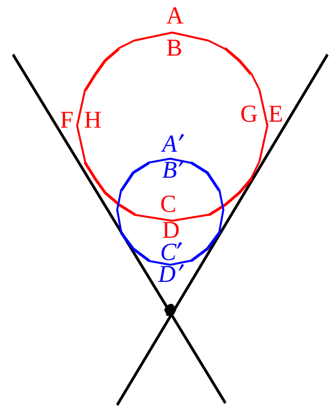
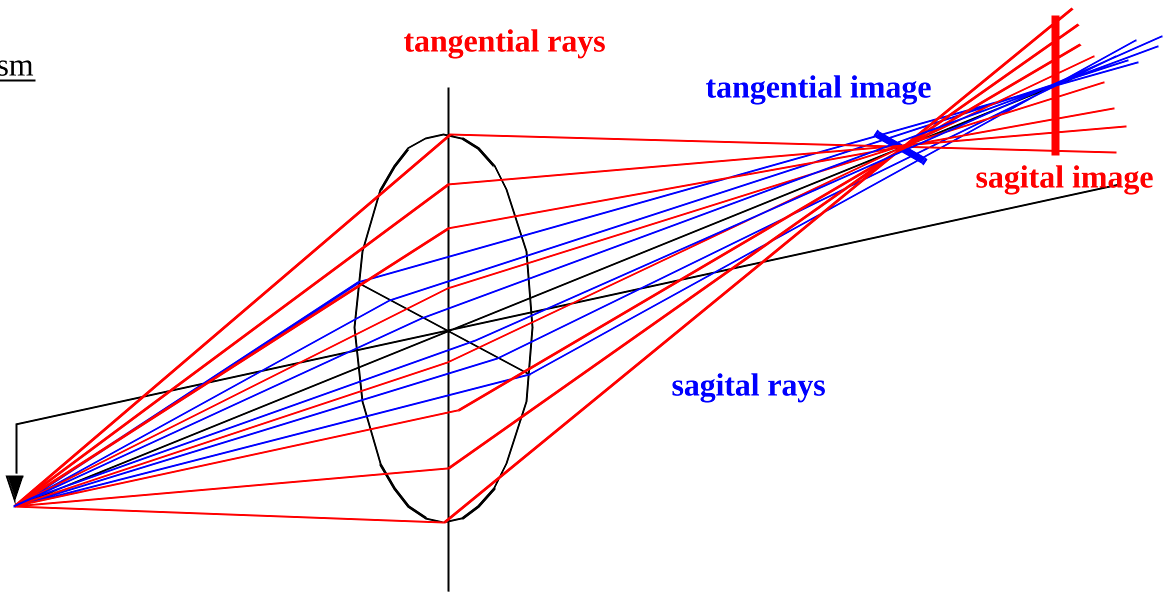


Image Plane

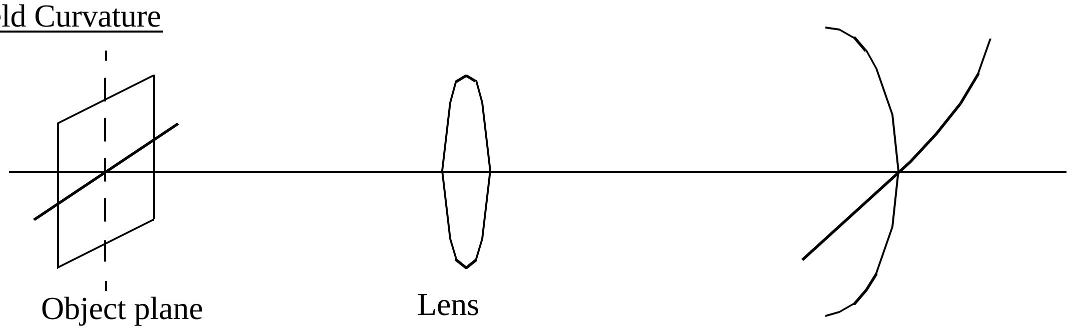


Astigmatism



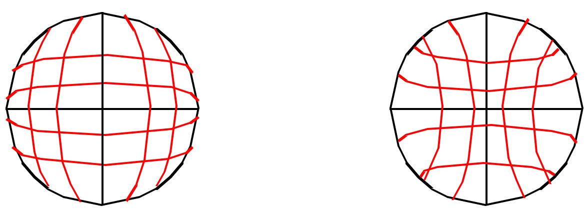
In astigmatism the tangential and sagittal images do not coincide. There are 2 line images with a circle of least confusion in the middle.

Field Curvature



Positive lenses give inward curvature
negative lenses give backward curvature.

Distortion: Field dependent magnification



Barrel distortion Pincushion distortion

Wave Front Aberration

In a wave-optics picture, the thin lens is represented by phase delay.

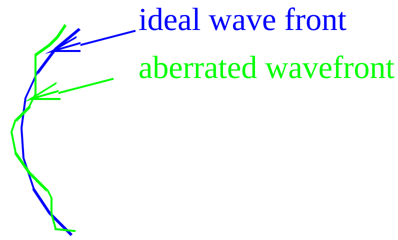
$$\phi(x, y) = -k\frac{x^2 + y^2}{2} = -k\Delta(x, y) \qquad (6.1)$$

Which gives Gaussian imaging. Aberrations modify ϕ . A spherical lens only gives this ϕ in the paraxial approximation.

We collect these phase errors at the exit pupil and use them to define a generalized pupil function:

(6.2)

W(x,y) : path length error



Expressed in this way, the primary aberrations are written as follows, with ρ: normalized radial coordinate in the pupil, and h': the image height

Spherical aberration:

Coma:

Astigmatism:

Field Curvature: $A_d \rho^2 h'^2$

Distortion: $A_t h'^3 \rho \cos\theta$

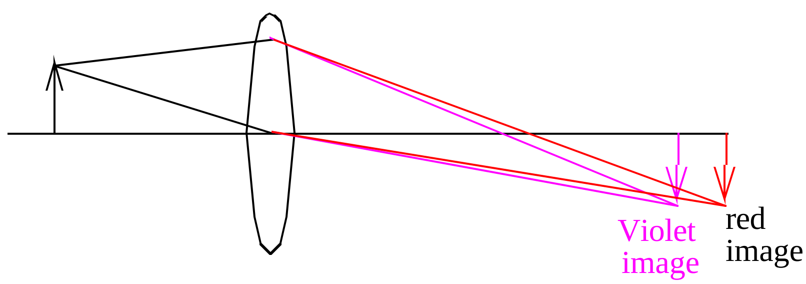
<u>Monochromatic Aberrations:</u> All of the preceding discussion refers to aberrations that do not depend on wavelength.

<u>Chromatic Aberrations:</u> Dependance of wavefront on wavelength.

Consider the simple thin lens equation:

(6.3)

The index n is generally λ dependent, $n(\lambda)$, so f is λ dependent.



Change in image distance: longitudinal chromatic aberration

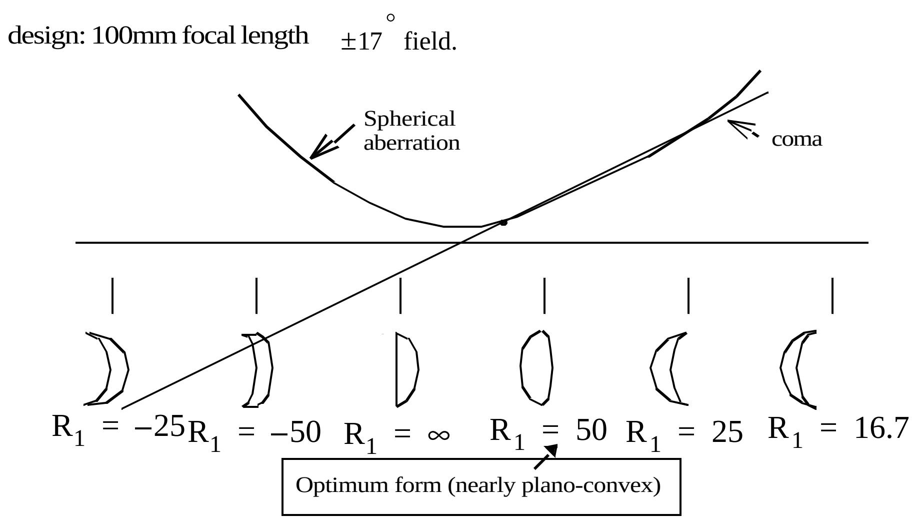
Change in magnification: lateral color. Lateral color is usually more noticeable

Achromat: lens designed to cancel chromatic aberration.

Lens Design:

- The general problem of lens design involves cancelling aberrations
- Aberration depends on the lens index, as well as the surface radii.
- Complex lens systems can minimize aberrations

Simple singlet case: For a given desired focal length, there is freedom to choose one of the radii for a singlet. The spherical aberration and coma depend on the particular choice, so these aberrations can be minimized by the design form. This is illustrated in the following diagram:



Fourier-space treatment of aberrations

Recall that the ideal effect of the lens is to impart a quadratic phase shift, multiplied by the aperture function. The amplitude impulse response: *h(u,v)* is a scaled Fourier transform of the pupil function. This can be generalized to the Fourier transform of $\mathcal{P}(x,y)$.

$$h(u, v) = \frac{1}{\lambda^2 z_o z_i} \int_{-\infty}^{\infty}\int \mathcal{P}(x, y) \exp\left[-j\frac{2\pi}{\lambda z_i}(ux + vy)\right] dx dy \tag{6.4}$$

$$= \frac{1}{\lambda^2 z_o z_i} \iint P(x, y) \exp[jkW(x, y)] \exp\left[-j\frac{2\pi}{\lambda z_i}(ux + vy)\right] dx dy \tag{6.5}$$

The amplitude transfer function is still the Fourier transform of *h,* so

$$H(f_x, f_y) = \mathcal{P}(-\lambda z_i f_x, -\lambda z_i f_y)$$

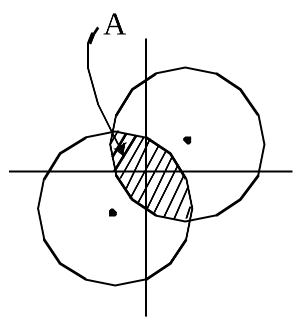
$$= P(\lambda z_i f_x, \lambda z_i f_y) \exp[jkW(-\lambda z_i f_x, -\lambda z_i f_y)] \qquad (6.6)$$

> Aberrations introduce a pure phase distortion within an unaffected passband: $P(\lambda z_i f_x, \lambda z_i f_y)$

The effects with coherent illumination are described by *H*. For incoherent illumination - we must consider the optical transfer function: $\mathcal{H}(f_x, f_y)$. Recall that the optical transfer function for a diffraction limited case is the normalized overlap area of displaced pupils.

Define $A(f_x, f_y)$ as the overlap area of

and (6.7)



Then for a diffraction limited system, the optical transfer function is

$$\mathcal{H}(f_x, f_y) = \frac{\int\int_{A(f_x, f_y)} dx dy}{\int\int_{A(0,0)} dx dy} \qquad (6.8)$$

With aberrations, the optical transfer function becomes:

$$\mathcal{H}(f_x, f_y) = \frac{\int\int_{A(f_x, f_y)} \exp\left\{jk\left[W\left(x + \frac{\lambda z_i f_x}{2}, y + \frac{\lambda z_i f_y}{2}\right) - W\left(x - \frac{\lambda z_i f_x}{2}, y - \frac{\lambda z_i f_y}{2}\right)\right]\right\} dx dy}{\int\int_{A(0,0)} dx dy} \qquad (6.9)$$

The aberrations can only decrease the modulation transfer function. It is easy to show that:

$$|\mathcal{H}(f_x, f_y)|^2_{\text{aberrations}} \le |\mathcal{H}(f_x, f_y)|^2_{\text{diffraction limited}} \qquad (6.10)$$

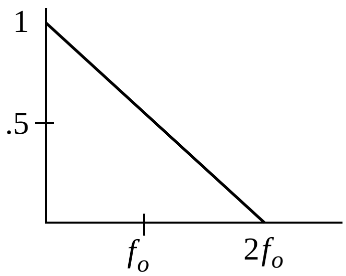
These aberrations do not reduce the absolute cutoff frequency, but they do reduce the contrast at some frequencies.

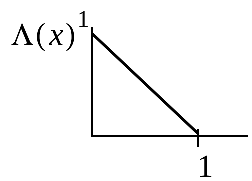
For a diffraction limited system, the optical transfer function is positive definite. With aberrations, the optical transfer function can be negative in certain frequency bands, resulting in a contrast reversal.

Calculation of the optical transfer function generally requires numerical calculation. An analytical solution is possible for a simple focal error. We need to assume a square aperture.

The diffraction limited OTF for a square aperture with a side 2w is:

$$\mathcal{H}(f_x, f_y) = \Lambda\left(\frac{f_x}{2f_o}\right)\Lambda\left(\frac{f_y}{2f_o}\right);\ f_o \equiv \frac{w}{\lambda z_i} \quad : \text{ cutoff of coherent system}$$





With a focal error, the phase at the pupil is quadratic, but with a radius of $z_a \neq z_i$. It does not converge exactly to the image plane.

$$\text{ideal phase } \phi(x, y) = \frac{\pi}{\lambda z_i}(x^2 + y^2)$$

$$\text{actual phase } \phi_a(x, y) = \frac{\pi}{\lambda z_a}(x^2 + y^2)$$

(6.11)

so

$$W(x, y) = \frac{1}{2}\left(\frac{1}{z_a} - \frac{1}{z_i}\right)(x^2 + y^2) \tag{6.12}$$

At the edge of the aperture along the x or y axis

$$W(w, 0) \equiv W_m = \frac{1}{2}\left(\frac{1}{z_a} - \frac{1}{z_i}\right)w^2 \tag{6.13}$$

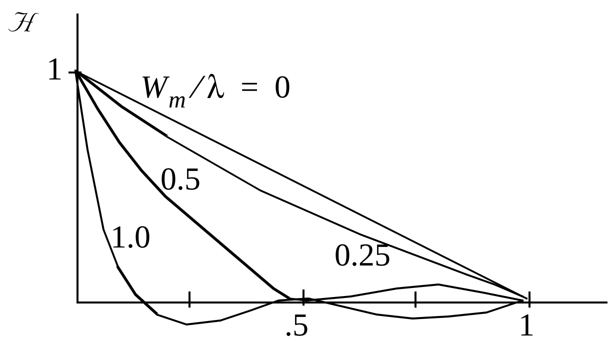
(6.14)

W_m is the measure of the magnitude of error.

The analytical solution for the OTF is then:

$$H(f_x, f_y) = \Lambda\left(\frac{f_x}{2f_o}\right)\Lambda\left(\frac{f_y}{2f_o}\right)\mathrm{sinc}\left[\frac{8W_m}{\lambda}\left(\frac{f_x}{2f_o}\right)\left(1 - \frac{|f_x|}{2f_o}\right)\right]\mathrm{sinc}\left[\frac{8W_m}{\lambda}\left(\frac{f_y}{2f_o}\right)\left(1 - \frac{|f_y|}{f_o}\right)\right] \quad (6.15)$$

Parametrized in W_m/λ, the number of waves of the path length difference:



Higher order aberrations

Most common systems are rotationally symmetric with a circular pupil. We can use polar coordinates for pupil points: $W(x, y) = W(r\cos\theta, r\sin\theta)$. The aberration function W depends on the object height, h (or image height h') and the pupil coordinates (r, θ).

If we use a normalized radius, and suppress the explicit dependence on image height h',

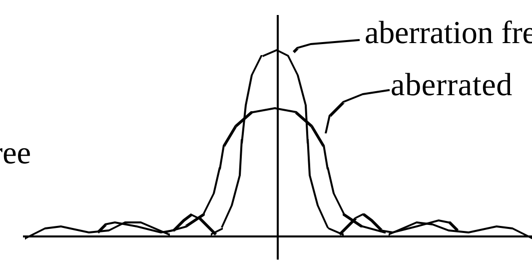
$$W(\rho, \theta) = A_s\rho^4 + A_c\rho^3\cos\theta + A_a\rho^2\cos^2\theta + A_d\rho^2 + A_t\rho\cos\theta \quad (6.16)$$

A_d : defocus A_t : tilt

The field curvature and distortion are equivalent to the field dependent focus and tilt errors, respectively.

A commonly used figure of merit: the Strehl ratio

$$S \equiv \frac{|h(0,0)|^2}{|h(0,0)|} \quad \begin{matrix}\text{aberrated} \\ \text{aberration-free}\end{matrix}$$



Define $\rho = r/w$; $\Phi(\rho, \theta) = \frac{2\pi}{\lambda}W(\rho, \theta)$

$$S = \frac{1}{\pi^2}\left|\int_0^1\int_0^{2\pi} \exp[j\Phi(\rho,\theta)]\rho\, d\rho\, d\theta\right|^2 \tag{6.17}$$

For small aberrations, expand $\exp(j\Phi) \cong 1 + j\Phi - \frac{1}{2}\Phi^2$

Then

$$S = \frac{1}{\pi^2}\left|\int_0^1\int_0^{2\pi}\left(1 - \frac{1}{2}\Phi^2 + j\Phi\right)\rho\, d\rho\, d\theta\right|^2 \tag{6.18}$$

$$= \left[1 - \frac{1}{2\pi}\int_0^1\int_0^{2\pi}\Phi^2\rho\, d\rho\, d\theta\right]^2 + \frac{1}{\pi^2}\left[\int_0^1\int_0^{2\pi}\Phi\rho\, d\rho\, d\theta\right]^2 \tag{6.19}$$

$$S \cong 1 - \frac{1}{\pi}\int_0^1\int_0^{2\pi}\Phi^2\rho\, d\rho\, d\theta + \frac{1}{\pi^2}\left[\int_0^1\int_0^{2\pi}\Phi\rho\, d\rho\, d\theta\right]^2 \tag{6.20}$$

Define the wavefront variance σ_W^2

$$\sigma_W^2 = \frac{1}{\pi}\int_0^1\int_0^{2\pi}[W(\rho,\theta) - W_{av}]^2\rho\, d\rho\, d\theta \tag{6.21}$$

$$= \frac{1}{\pi}\int_0^1\int_0^{2\pi}W^2(\rho,\theta)\rho\, d\rho\, d\theta - W_{av}^2 \tag{6.22}$$

with

$$W_{av} = \frac{1}{\pi}\int_0^1\int_0^{2\pi}W(\rho,\theta)\rho\, d\rho\, d\theta \tag{6.23}$$

Thus, we can write

(6.24)

The Strehl ratio depends only on σ_ϕ^2 - not on the details of aberration function. This leads us to the idea of balanced aberrations.

Example: we can balance spherical aberration with defocus

$$\Phi(\rho) = A_s\rho^4 + A_d\rho^2 \tag{6.25}$$

We choose just the right amount of defocus to minimize the wavefront variance:

$$\qquad , \tag{6.26}$$

which leads to $A_d = -A_s$.

<u>Balanced aberration</u>: An aberration of a certain order in a power expansion in pupil coordinates is mixed with lower order aberrations to minimize σ_ϕ^2.

<u>Zernike circle polynomials</u> *See Handout: Mahajan, Appl. Opt. 33, 8124 (1994). For further detail: Mahajan, "Aberration theory made simple" SPIE press, 1991*

•Balanced

•Orthonormal over a unit circle

•Complete set

$$\phi(\rho, \theta) = kW(\rho, \theta) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\{\varepsilon_m\sqrt{2(n+1)}R_n^m(\rho)[C_{nm}\cos m\theta + S_{nm}\sin m\theta]\} \tag{6.27}$$

$$\qquad \tag{6.28}$$

where *n* and *m* are positive integers, with $n - m \geq 0$, and always even.

The radial function :

$$R_n^m(\rho) = \sum_{s=0}^{(n-m)/2}\frac{(-1)^s(n-s)!}{s![(n+m)/2-s]![(n-m)/2-s]!}\rho^{n-2s} \tag{6.29}$$

Wavefront variance:

$$\langle W^2(\rho, \theta)\rangle = \sum_n\sum_m[C_{nm}^2 + S_{nm}^2] \tag{6.30}$$

The Zernike polynomials are typically given an ordering. However, be careful! The ordering is not universally agreed to. Different texts and even different lens design and optical imaging analysis software use their own ordering convention.

$$W(\rho, \theta) = \sum_{j=1}^{\infty} a_j Z_j(\rho, \theta) \tag{6.31}$$

$$Z_{even\,j} = \sqrt{2(n+1)} R_n^m(\rho) \cos m\theta \qquad m \neq 0 \tag{6.32}$$

$$Z_{odd\,j} = \sqrt{2(n+1)} R_n^m(\rho) \sin m\theta \qquad m \neq 0 \tag{6.33}$$

$$Z_j = \sqrt{n+1} R_n^m(\rho) \qquad m = 0 \tag{6.34}$$

The table in the handout gives one of the possible orderings $\{j, n, m\}$

Point spread functions for Zernike's (circular pupils)

We write the aberrated pupil function inside the unit circle as $\mathcal{P}(\rho, \theta) = e^{i\phi(\rho, \theta)}$ $\rho \equiv r/w$

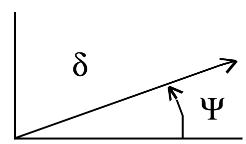
$$\phi(\rho, \theta) = kW(\rho, \theta) \quad \text{phase aberration}$$

Expand $\phi(\rho, \theta)$ in Zernike polynomials

(6.35)

We wish to calculate the aberrated point spread function: [object radial coordinates: (δ, Ψ)]

Take the Fourier transform of $P(\rho, \theta)$:



$$h(\delta, \Psi) = \int_0^1 \int_0^{2\pi} e^{j\phi(r,\theta)} e^{j2\pi r\delta\cos(\theta-\Psi)} r\,dr\,d\theta$$

If the aberration is small $\phi \ll 2\pi$, $e^{j\phi} \cong 1 + j\phi + \dots$

$$1 + j\phi = \sum_n \sum_m \alpha'_{nm} R_n^m \begin{Bmatrix} \cos m\theta \\ \sin m\theta \end{Bmatrix}, \tag{6.36}$$

where

$$\alpha'_{nm} = j\alpha_{nm} \qquad n \neq 0 \tag{6.37}$$

$$\alpha'_{oo} = 1 + j\alpha_{oo} \qquad n = 0 \tag{6.38}$$

$$h(\delta, \Psi) = \sum_n \sum_m \int_0^1 \int_0^{2\pi} \alpha'_{nm} R_n^m(\rho) \begin{Bmatrix} \cos m\theta \\ \sin m\theta \end{Bmatrix} e^{j2\pi r\delta\cos(\theta-\Psi)} r\,dr\,d\theta \tag{6.39}$$

It can be shown: (see Born and Wolf)

$$h(\delta, \Psi) = \sum_n \sum_m \alpha'_{nm} \pi e^{im\frac{\pi}{2}} + (-1)^{\frac{n-m}{2}} \left(\frac{2J_{n+1}(2\pi\delta)}{2\pi\delta}\right) \begin{Bmatrix} \cos m\Psi \\ \sin m\Psi \end{Bmatrix} \qquad (6.40)$$

The point spread functions for Zernike aberrations are related to higher order Bessel functions.