Chapter 8
Ultimate Coherence In The Quantum Limit


**Question:** What happens in Michelson interferometer when the source intensity is so low that only one photon is in the apparatus at a time?

**Answer:** It depends on the details of the quantum state of the radiation field.

We know from quantum theory of the radiation field that each mode (usually defined by quantizing in a box that imposes boundary conditions on the field) has quantized energy. There exists an infinite number of photon modes, labelled by $\hat{k}$.

$$\omega, k : \text{ usual classical definitions}$$

$$n_k : \text{ the number of photons in mode } k \text{ must be an integer } n = 0, 1, 2, \ldots$$

**Number states:** well defined photon numbers

$$|n_k, n_k, n_k, \ldots\rangle = |n_k\rangle|n_{k_2}\rangle|n_{k_3}\rangle\ldots$$

**Creation and destruction operators:**

$$\text{destruction: } \hat{a}_k|n_k\rangle = n_k^{1/2}|n_k - 1\rangle$$

$$\text{creation: } \hat{a}_k^\dagger|n_k\rangle = (n_k + 1)^{1/2}|n_k + 1\rangle$$

**Number operator**

$$\hat{n}_k \equiv \hat{a}_k^\dagger \hat{a}_k$$

It is easy to show that $\hat{n}_k|n_k\rangle = \hat{a}_k^\dagger \hat{a}_k|n_k\rangle = n_k|n_k\rangle$

**Hamiltonian for radiation field:**

$$\langle \varepsilon \rangle = \sum_k \langle \varepsilon_k \rangle = \sum_k \hbar \omega_k \left( n_k + \frac{1}{2} \right)$$

Number states are eigenstates of the Hamiltonian with well defined energy.
Quantum mechanical phase

Classical electrical field with mode $\vec{k}$:

$$E = \frac{1}{2} E_o \{ \exp(-j\omega t + j\vec{k} \cdot \vec{r} + j\phi) + \exp(j\omega t - j\vec{k} \cdot \vec{r} - j\phi) \} \tag{8.8}$$

Quantum mechanical electric field operator:

$$E = j(\hbar \omega / (2\varepsilon_0 V))^{1/2} \{ \hat{a} \exp(-j\omega t + j\vec{k} \cdot \vec{r}) + \hat{a}^\dagger \exp(j\omega t - j\vec{k} \cdot \vec{r}) \} \tag{8.9}$$

We can express $\hat{a}$ as product of amplitude and phase operators. There is no unique definition of the quantum mechanical phase; however, this should have the same significance as classical phase in the classical limit. Phase is a genuine physical observable so it must be represented by a Hermitian operator.

The usual definition of the phase operators are:

$$\hat{a} = \left( \hat{n} + \frac{1}{2} \right) \exp(j\phi) \tag{8.10}$$

$$\hat{a}^\dagger = \exp(-j\phi) \left( \hat{n} + \frac{1}{2} \right)^{1/2} \tag{8.11}$$

where the actual phase operators themselves are:

$$\exp(\hat{j}\phi), \exp(-\hat{j}\phi) \tag{8.12}$$

Note: these are operators, where the order of operation matters!

These are not Hermitian operators (not observable), but the combinations:

$$\cos \phi = \frac{1}{2} \{ \exp(\hat{j}\phi) + \exp(-\hat{j}\phi) \} \tag{8.13}$$

$$\sin \phi = \frac{1}{2j} \{ \exp(\hat{j}\phi) + \exp(-\hat{j}\phi) \} \tag{8.14}$$

are Hermitian and represent observables.

The number operator and phase operators do not commute.

$$[\hat{n}, \cos \phi] = -j\sin \phi \tag{8.15}$$

$$[\hat{n}, \sin \phi] = j\cos \phi \tag{8.16}$$

Significance: there are no simultaneous eigenstates of number and phase and there is an equivalent Heisenberg uncertainty principle for number and phase:

$$\Delta n \Delta \phi \geq \frac{\hbar}{2} \tag{8.17}$$
For a particular state of the field, $\Delta n$, $\Delta \cos \phi$, $\Delta \sin \phi$ refer to the rms deviation of series of measurements. $\langle \sin \phi \rangle$, $\langle \cos \phi \rangle$ refer to the expectation value of the operator for a given state of field.

For number states $\Delta n = 0$

$$\langle n | \cos \phi | n \rangle = \langle n | \sin \phi | n \rangle = 0 \quad (8.19)$$

but

$$\langle n | (\cos \phi)^2 | n \rangle = \langle n | (\sin \phi)^2 | n \rangle = \begin{cases} \frac{1}{2} & n \neq 0 \\ \frac{1}{4} & n = 0 \end{cases} \quad (8.20)$$

Except for $n = 0$,

$$\Delta \cos \phi = \Delta \sin \phi = \frac{1}{\sqrt{2}} \quad (8.21)$$

This corresponds to the case of a classical phase angle being uniformly distributed $[0, 2\pi]$

Thus, number states have a complete uncertain phase, but a perfectly defined amplitude.

Superpositions of number states:

We can construct states $|\phi\rangle = \sum_n c_n |n\rangle$ that are eigenstates of phase with zero uncertainty in $\cos \phi$, $\sin \phi$. These have infinite uncertainty in $n$. This leads to infinite $\Delta \varepsilon$. This is closely related to the usual $\Delta \varepsilon \Delta t \geq \hbar$ uncertainty. The radiation field with a perfectly certain phase has infinite energy uncertainty. This is not physical.

Minimum uncertainty state:
\[ |\alpha\rangle \equiv \exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_n \frac{\alpha^n}{(n!)^{1/2}}|n\rangle \]  

(8.22)

This is also called a “coherent” state.

The probability of finding \( n \) photons in node \( \alpha \) is

\[ |\langle n|\alpha\rangle|^2 = \exp(-|\alpha|^2)\frac{|\alpha|^{2n}}{n!}, \]  

(8.23)

a Poisson probability distribution. The mean is \( |\alpha|^2 \), \( \Delta n = |\alpha| \).

1. In a limit of high excitation, these come closest to a classical wave of stable amplitude and fixed phase.
2. A single mode laser, well above the threshold can be shown to generate a coherent state excitation.
3. Coherent states are eigenstates of \( \hat{a} \):

\[ \hat{a}|\alpha\rangle = \alpha|\alpha\rangle \]  

(8.24)

For \( \alpha = |\alpha|e^{i\theta} \), it can be shown that for coherent states \( \Delta n \Delta \cos \phi = \frac{1}{2} \sin \theta \) for large \( |\alpha| \).

This reaches a minimum uncertainty product

Examples

\[ E(t) \quad |\alpha|^2 = 4 \quad |\alpha|^2 = 40 \quad |\alpha|^2 = 400 \]

Finally, we can answer the question posed at the outset concerning the result in a Michelson interferometer. For low intensities, the photon counting rate by \( D \equiv \bar{m} \) is the signal. Due to uncertainties discussed above, \( \bar{m} \) has quantum fluctuations. A quantum analysis of the Michelson
interferometer shows the following (near $\tau = 0$)

\[ \Delta m = (m)^{1/2} \left( 1 - \frac{\bar{m}}{n} \right)^{1/2} \]

\[ \Delta m = (\bar{m})^{1/2} \]

quantum noise

coherent state

Poisson counting statistics