Chapter 9
Spatial Coherence

[Continue reading: Goodman, Statistical Optics, Chapter 5]

Again we use the interference experiment as a vehicle for discussing spatial coherence

Young’s double slit experiment

We expect an interference fringe at Q under two conditions:

a) The time delay difference is less than the coherence time of source

\[(9.1)\]

b) The visibility of the fringe depends on the degree of correlation between light waves at the two pinholes. Here the source size plays a big role.

To see a relation between source size and the degree of correlation, we assume a narrowband, point source $S_1$

$S_1$ produces a sinusoidal fringe, with high contrast. Now add a second source, $S_2$, nearby, but radiating independently (not coherent with $S_1$). $S_2$ produces its own fringe, shifted spatially from the $S_1$ fringe. The fringe period depends on the $P_1$-$P_2$ separation. The shift depends on the $S_1$-$S_2$ separation.
When \( P_1 \) and \( P_2 \) are far apart, the fringe period becomes smaller than the shift. This results in cancellation. When the source actually consists of many such independent point sources (an extended source), then the fringes wash out.

For a given source size, the fringe visibility depends on the spacing of \( P_1-P_2 \). This gives a measure of spatial coherence at screen 1. The fringe visibility falls off as the \( P_1-P_2 \) separation falls outside of the coherence area.

**Analytical treatment**

A generalization of the Huygens-Fresnel principle to broadband and narrowband cases (review Goodman, *Fourier Optics*, Section 3.8).

Using the inverse Fourier transform:

\[
S_1 \quad S_2 \quad P_1 \quad P_2 \quad \text{fringe} \quad \text{resultant fringe} \quad S_2 \text{ fringe} \quad S_1 \text{ fringe}
\]

(9.2)

(9.3)

Let \( \nu' = -\nu \), then
Expressed this way, $u(P_1, t)$, $u(P_o, t)$ are linear combinations of monochromatic waves as originally expressed at the beginning of the course. The idea is to use the Huygens-Fresnel integral to relate $U(P_o, \nu')$ to $U(P_1, \nu')$ for each $\nu'$ (using the monochromatic result), then to use the above integral superposition to find $u(P_o, t)$ in terms of $u(P_1, t)$.

**Huygens-Fresnel integral:**

$$U(P_o, \nu') = -j \frac{\nu'}{c} \int \int U(P_1, \nu') \frac{\exp(j2\pi \nu'r_{01}/c)}{r_{01}} \cos(n, r_{01}) dS$$  \hspace{1cm} (9.6)

Insert this equation into equation (9.5) above:

$$u(P_o, t) = -\int \int \cos(n, r_{01}) \int_\Sigma j2\pi \nu' U(P_1, \nu') \exp[-j2\pi \nu'(t - \frac{r_{01}}{c})] d\nu' dS$$  \hspace{1cm} (9.7)

Differentiate equation (9.4)

$$\frac{d}{dt} u(P_1, t) = -\int j2\pi \nu' U(P_1, \nu') \exp(-j2\pi \nu't) d\nu'$$  \hspace{1cm} (9.8)

We can then use this to write

$$\frac{d}{dt} u(P_1, t) = -\int j2\pi \nu' U(P_1, \nu') \exp(-j2\pi \nu't) d\nu'$$  \hspace{1cm} (9.9)

This is a generalized Huygens-Fresnel integral for broadband light.

For narrowband light, $\Delta \nu \ll \nu$, re-examine equation (9.7). The width of $U(P, \nu')$ is $\Delta \nu$, so we can take the $\nu'$ factor out of the integral as $c/\lambda$.

Then equation (9.7) can be written:
Using (9.2) this is $u(P_1, t - r_{01}/c)$

So,

This is the Huygens-Fresnel integral for *narrowband* light.

**Application to Young’s experiment**

We have assumed pinholes small enough so that the field is constant over the whole pinhole. $K_1$ and $K_2$ are complex constants. (Pure imaginary)

\[
I(Q) = |K_1|^2 \langle |u(P_1, t - r_1/c)|^2 \rangle + |K_2|^2 \langle |u(P_2, t - r_2/c)|^2 \rangle + K_1 K_2^* \langle u(P_1, t - r_1/c) u^*(P_2, t - r_2/c) \rangle + \text{complex conjugate} \tag{9.14}
\]

Write

\[
I^{(1)}(Q) = |K_1|^2 \langle |u(P_1, t - r_1/c)|^2 \rangle \quad \text{intensity at } Q \text{ with } P_2 \text{ blocked.} \tag{9.15}
\]
\[ I^{(2)}(Q) = |K_2|^2 \left| \mu(P_2, t - r_2/c) \right|^2 \] intensity at \( Q \) with \( P_1 \) blocked. \hfill (9.16)

We then write the cross correlation of the light at \( P_1 \) with \( P_2 \) as:

\[ I(Q) = I^{(1)}(Q) + I^{(2)}(Q) + K_1 K_2^* \Gamma_{12} \left( \frac{r_2 - r_1}{c} \right) + K_1^* K_2 \Gamma_{21} \left( \frac{r_1 - r_2}{c} \right) \] \hfill (9.18)

Since \( K_1, K_2 \) are pure imaginary, \( K_1 K_2^* = K_1^* K_2 = \kappa_1 \kappa_2 \), and \( \kappa_1 = |K_1| \kappa_2 = |K_2| \), also \( \Gamma_{21}(-\tau) = \Gamma_{12}^*(\tau) \).

\[ I(Q) = I^{(1)}(Q) + I^{(2)}(Q) + 2 \kappa_1 \kappa_2 \text{Re} \left\{ \Gamma_{12} \left( \frac{r_2 - r_1}{c} \right) \right\} \] \hfill (9.19)

Again we normalize the coherence function using the Schwarz inequality,

\[ |\Gamma_{12}(\tau)|^2 \leq |\Gamma_{11}(0)| \Gamma_{22}(0) \] note: \( \Gamma_{11}(0) = I(P_1) \) and \( \Gamma_{22}(0) = I(P_2) \) \hfill (9.20)

Where \( \Gamma_{11}(\tau), \Gamma_{22}(\tau) \) are self-coherence functions. These were defined in the discussion of the Michelson interferometer in chapter 7 of these notes.

So we can normalize \( \Gamma_{12}(\tau) \) by \( [\Gamma_{11}(0) \Gamma_{22}(0)]^{1/2} \).

\[ I^{(1)}(Q) = \kappa_1^2 \Gamma_{11}(0) \] \hfill (9.22)

\[ I^{(2)}(Q) = \kappa_2^2 \Gamma_{22}(0) \] \hfill (9.23)

Further simplifying the expression for the fringe pattern, we can write:

\[ I(Q) = I^{(1)}(Q) + I^{(2)}(Q) + 2 \sqrt{I^{(1)}(Q)I^{(2)}(Q)} \text{Re} \left\{ \gamma_{12} \left( \frac{r_2 - r_1}{c} \right) \right\} \] \hfill (9.24)

Anticipating the form of the fringe pattern, we write:

\[ \gamma_{12}(\tau) = \gamma_{12}(\tau) \exp \{-j[2\pi \nu \tau - \alpha_{12}(\tau)]\} \] \hfill (9.25)
I(Q) = I^{(1)}(Q) + I^{(2)}(Q) + 2\sqrt{I^{(1)}(Q)I^{(2)}(Q)}\gamma_{12}\left(\frac{r_2 - r_1}{c}\right)\cos\left[2\pi\nu\left(\frac{r_2 - r_1}{c}\right) - \alpha_{12}\left(\frac{r_2 - r_1}{c}\right)\right] \tag{9.26}

I^{(1)}(Q), I^{(2)}(Q) vary spatially as pinhole diffraction patterns. We assume these pinholes are very small so that the intensities are nearly constant over the observation region.

On top of this background, we have the fringe pattern, with:
- the period in space depends on \nu and the geometry \(r_2 - r_1\).
- a slowly varying envelope and phase modulation.

Near zero path difference \((r_2 - r_1 = 0)\), the fringe visibility is

\[\gamma_{12}(0)\]

\(\gamma_{12}(0)\) measures the coherence of \(u(P_1, t)\) and \(u(P_2, t)\).

- when \(P_1 = P_2\) \(\gamma_{11}(0) = 1\) the pinholes coincide

As \(P_1, P_2\) separate, \(\gamma_{12}(0)\) varies. This indicates the spatial coherence of the light striking the pinhole plane.

As the path difference \(\left(\frac{r_2 - r_1}{c}\right)\) varies, the fringe envelope tapers off. This is an indication of the temporal coherence effect.