2-3. \( \mathcal{F} \mathcal{F} \{g(x,y)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} df_x df_y \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\xi d\eta \ g(\xi, \eta) e^{-j2\pi (\xi f_x + \eta f_y)} \right\} e^{-j2\pi (f_x x + f_y y)} \).

Interchange the orders of integration, yielding

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\xi d\eta \ g(\xi, \eta) \exp\left\{ -j2\pi \left[ (\xi + x) f_x + (\eta + y) f_y \right] \right\}. \]

But the right-hand double integral is identically the same as \( \delta(\xi + x, \eta + y) \), and the sifting property can be applied to the remaining double integral,

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\xi d\eta \ g(\xi, \eta) \delta(\xi + x, \eta + y) d\xi d\eta = g(-x, -y). \]

The result for \( \mathcal{F}^{-1} \mathcal{F}^{-1}\{g(x,y)\} \) is derived the same way with a change of sign in both exponentials.

(b) The simplest method of proof is to show that

\[ \mathcal{F}^{-1} \{G(f_x, f_y) \otimes H(f_x, f_y)\} = g(x, y) \ h(x, y). \]

Remembering that the \( \mathcal{F}^{-1} \) operator operates on the variables \((f_x, f_y)\),

\[ \mathcal{F}^{-1} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\xi, \eta) H(f_x - \xi, f_y - \eta) d\xi d\eta \right\} \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\xi, \eta) \mathcal{F}^{-1}\{H(f_x - \xi, f_y - \eta)\} d\xi d\eta \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\xi, \eta) \exp[j2\pi(\xi x + \eta y)] d\xi d\eta \ h(x, y) \]

\[ = g(x, y) \ h(x, y) \]

where the shift theorem for inverse transforms has been used.

(c) \( \mathcal{F}\{\nabla^2 g(x,y)\} = \mathcal{F}\left\{ \frac{\partial^2}{\partial x^2} g(x,y) + \frac{\partial^2}{\partial y^2} g(x,y) \right\} \). Now

\[ \mathcal{F}\left\{ \frac{\partial^2}{\partial x^2} g(x,y) \right\} = \frac{\partial^2}{\partial x^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(f_x, f_y) e^{j2\pi(f_x x + f_y y)} df_x df_y \]

\[ + \frac{\partial^2}{\partial y^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(f_x, f_y) e^{j2\pi(f_x x + f_y y)} df_x df_y. \]
\[
\begin{align*}
&= \iint_{-\infty}^{\infty} G(f_X, f_Y) \frac{\partial^2}{\partial x^2} e^{i2\pi(f_x x + f_y y)} \, df_X \, df_Y \\
&= \iint_{-\infty}^{\infty} \left[ -4\pi^2 \left( f_y^2 + f_y^2 \right) \right] G(f_X, f_Y) e^{i2\pi(f_x x + f_y y)} \, df_X \, df_Y.
\end{align*}
\]

We conclude that
\[
\mathcal{F}\left\{ \nabla^2 g(x, y) \right\} = -4\pi^2 \left( f_x^2 + f_y^2 \right) G(f_X, f_Y) = -4\pi^2 \left( f_x^2 + f_y^2 \right) \mathcal{F}\left\{ g(x, y) \right\}.
\]
2-5. Note that since \( G(f_X, f_Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) e^{-j2\pi(f_X x + f_Y y)} \, dx \, dy \), we see that

\[
G(0, 0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) \, dx \, dy.
\]

Similarly, since \( g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(f_X, f_Y) e^{j2\pi(f_X x + f_Y y)} \, df_X \, df_Y \), we have

\[
g(0, 0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(f_X, f_Y) \, df_X \, df_Y.
\]
Thus

\[ \Delta_{XY} = \left| \frac{\iint_{-\infty}^{\infty} g(x, y) \, dx \, dy}{g(0, 0)} \right| = \left| \frac{G(0, 0)}{g(0, 0)} \right| = \left| \frac{G(0, 0)}{\iint_{-\infty}^{\infty} G(f_x, f_y) \, df_x \, df_y} \right| = \frac{1}{\Delta_{f_x f_y}}. \]

Hence \( \Delta_{xy} \Delta_{f_x f_y} = 1. \)
2.8. To avoid confusion, let's call the frequencies of the applied cosinusoidal signal \((\bar{f}_X, \bar{f}_Y)\). Note that the input can be expanded into a sum of two complex exponentials,

\[
g(x, y) = \cos[2\pi(\bar{f}_X x + \bar{f}_Y y)] = \frac{1}{2} \exp[j2\pi(\bar{f}_X x + \bar{f}_Y y)] + \frac{1}{2} \exp[-j2\pi(\bar{f}_X x + \bar{f}_Y y)].
\]

Now to have any hope of producing a cosine at the output, we had better insist that the system be invariant, for only then can we expect the exponential nature of the two input components to be preserved. For an invariant system, each complex-exponential input produces a complex-exponential output of the same frequency, but with a possible change of amplitude and phase, as determined by the transfer function. Remembering that the complex exponentials are eigenfunctions of linear, invariant systems, we write the output \(v(x, y)\) as

\[
v(x, y) = \frac{1}{2} H(\bar{f}_X, \bar{f}_Y) \exp[j2\pi(\bar{f}_X x + \bar{f}_Y y)] + \frac{1}{2} H(-\bar{f}_X, -\bar{f}_Y) \exp[-j2\pi(\bar{f}_X x + \bar{f}_Y y)],
\]

where \(H(f_X, f_Y)\) is the transfer function of the system, given by the Fourier transform of the impulse response \(h(x, y)\). The transfer function can be written as the product of an amplitude function and a phase function,

\[
H(f_X, f_Y) = A(f_X, f_Y) e^{i\phi(f_X, f_Y)},
\]
where $A(f_X, f_Y) \geq 0$. Thus the output can be written

$$v(x, y) = \frac{1}{2} A(f_X, f_Y) \exp[j2\pi(f_X x + f_Y y) + \phi(f_X, f_Y)]$$

$$+ \frac{1}{2} A(-f_X, -f_Y) \exp[-j2\pi(f_X x + f_Y y) + \phi(-f_X, -f_Y)]$$

Now we ask under what conditions can the above two exponential terms be combined to form a cosinusoidal output of frequency $(f_X, f_Y)$? The answer is that the following two conditions must be satisfied:

$$A(-f_X, -f_Y) = A(f_X, f_Y)$$

$$\phi(-f_X, -f_Y) = -\phi(f_X, f_Y),$$

i.e. the magnitude of the transfer function must be even and the phase must be odd. These symmetry relations will be satisfied if and only if the impulse response of the system, $h(x, y)$, is real-valued. Thus, to summarize, the required conditions are that the system be linear and invariant, and that its impulse response be real-valued.
2-11. (a) By the convolution theorem,

\[ P(f_X, f_Y) = G(f_X, f_Y) \cdot XY \text{comb}(X f_X) \text{comb}(Y f_Y) \]
where we have used the similarity theorem and the fact that the Fourier transform of a comb function is another comb function. Further simplification results from the following relation:

\[
XY \text{comb}(X f_X) \text{comb}(Y f_Y) = XY \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \delta(X f_X - n, Y f_Y - m)
\]

\[
= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \delta(f_X - \frac{n}{X}, f_Y - \frac{m}{Y}),
\]

where we have used the fact that \(\delta(ax, by) = \frac{1}{|a|b} \delta(x, y)\). We have assumed in the above that \(X \geq 0, Y \geq 0\).

(b) The Fourier transform of the given \(g(x, y)\) is found as follows:

\[
\mathcal{F}\{g(x, y)\} = \frac{XY}{4} \text{sinc} \left(\frac{X}{2} f_X\right) \text{sinc} \left(\frac{Y}{2} f_Y\right),
\]

where the similarity theorem has been used. The figure below shows sketches of \(g(x, 0)\) and \(p(x, 0)\) in this case.

Figure 2-11:
2-13. The object \( U_o(x, y) \) has a band-unlimited spectrum, while the transfer function \( H(f_X, f_Y) \) of the system is bandlimited to the region \( |f_X| \leq B_X, |f_Y| \leq B_Y \). Because of the bandlimitation on \( H \), it is possible to write

\[
H(f_X, f_Y) = H(f_X, f_Y) \text{rect} \left( \frac{f_X}{2B_X} \right) \text{rect} \left( \frac{f_Y}{2B_Y} \right).
\]

Since the imaging system is both linear and invariant, the image and object spectra, \( G_i \) and \( G_o \), respectively, can be related by

\[
G_i(f_Y, f_Y) = H(f_X, f_Y) G_o(f_X, f_Y) = H(f_X, f_Y) \left[ \text{rect} \left( \frac{f_X}{2B_X} \right) \text{rect} \left( \frac{f_Y}{2B_Y} \right) G_o(f_X, f_Y) \right].
\]

From this equation we can see directly that the output spectrum can be viewed as resulting from the application of a new fictitious object with spectrum

\[
G'_o(f_X, f_Y) = \text{rect} \left( \frac{f_X}{2B_X} \right) \text{rect} \left( \frac{f_Y}{2B_Y} \right) G_o(f_X, f_Y).
\]

In the space domain, the relation between the fictitious object and the actual object is

\[
U'_o(x, y) = U_o(x, y) \otimes 4B_X B_Y \text{sinc}(2B_X x) \text{sinc}(2B_Y y)
\]

\[
= 4B_X B_Y \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U_o(\xi, \eta) \text{sinc} [2B_X (x - \xi)] \text{sinc} [2B_Y (y - \eta)] \, d\xi \, d\eta.
\]

Since \( U'_o \) is bandlimited, it can be reconstructed from samples taken at the Nyquist rate, i.e. samples taken at coordinates \( x_n = \frac{n}{2B_X}, y_m = \frac{m}{2B_Y} \). The sampled object which will yield \( U'_o \)
after low pass filtering is given by

\[
\hat{U}'_o(x, y) = \text{comb}(2B_X x) \text{comb}(2B_Y y) U'_o(x, y)
= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} U'_o \left( \frac{n}{2B_X}, \frac{m}{2B_Y} \right) \delta \left( x - \frac{n}{2B_X}, y - \frac{m}{2B_Y} \right).
\]

Substituting the expression derived above for \( U'_o \),

\[
\hat{U}'_o(x, y) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U_o(x, y) \text{sinc}(n - 2B_X \xi) \text{sinc}(m - 2B_Y \eta) \, d\xi \, d\eta \right]
\times \delta \left( x - \frac{n}{2B_X}, m - \frac{m}{2B_Y} \right).
\]

This array of point sources will yield the same image as the original object \( U_o(x, y) \).
3-5. Using Eq. (3-63) we have the following:

(a) Circular aperture of diameter $d$:

\[
A \left( \frac{\alpha}{\lambda}, \frac{\beta}{\lambda}; 0 \right) = \mathcal{B} \left\{ \text{circ} \left( \frac{2r}{d} \right) \right\}_{r_x=\alpha/\lambda, r_y=\beta/\lambda}
\]

using the similarity theorem for Fourier-Bessel transforms (Eq. (2-34)) and the Fourier-Bessel transform pair of Eq. (2-35),

\[
A \left( \frac{\alpha}{\lambda}, \frac{\beta}{\lambda}; 0 \right) = \frac{d^2}{4} \frac{J_1 \left( \frac{2\pi \rho d}{2} \right)}{\frac{d \rho}{2}} = \frac{d}{2} \frac{J_1 \left( \pi \rho d \right)}{\rho}.
\]

Finally, note that $\rho = \sqrt{f_X^2 + f_Y^2} = \sqrt{\left( \frac{\alpha}{\lambda} \right)^2 + \left( \frac{\beta}{\lambda} \right)^2}$ yielding

\[
A \left( \frac{\alpha}{\lambda}, \frac{\beta}{\lambda}; 0 \right) = \frac{d^2}{4} \frac{J_1 \left( \frac{2\pi \rho d}{2} \right)}{\frac{d \rho}{2}} = \frac{d}{2} \frac{J_1 \left( \pi \sqrt{\left( \frac{\alpha}{\lambda} \right)^2 + \left( \frac{\beta}{\lambda} \right)^2} \right)}{\sqrt{\left( \frac{\alpha}{\lambda} \right)^2 + \left( \frac{\beta}{\lambda} \right)^2}}.
\]

(b) A circular opaque disk of diameter $d$ can be modeled by the following amplitude transmittance function:

\[
t_A(x, y) = 1 - \text{circ} \left( \frac{2r}{d} \right).
\]

From the linearity theorem of Fourier analysis it follows that the angular spectrum of this structure is

\[
A \left( \frac{\alpha}{\lambda}, \frac{\beta}{\lambda}; 0 \right) = \delta \left( \frac{\alpha}{\lambda}, \frac{\beta}{\lambda} \right) - \frac{d}{2} \frac{J_1 \left( \pi \sqrt{\left( \frac{\alpha}{\lambda} \right)^2 + \left( \frac{\beta}{\lambda} \right)^2} \right)}{\sqrt{\left( \frac{\alpha}{\lambda} \right)^2 + \left( \frac{\beta}{\lambda} \right)^2}}.
\]
3-7. (a) Substituting \( U(x, y, z) \approx A(x, y, z)e^{j kz} \) into the Helmholtz equation \((\nabla^2 + k^2)U = 0\),

\[
\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2 \right] A(x, y, z)e^{j kz} = 0.
\]

Then,

\[
\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] Ae^{j kz} + \frac{\partial}{\partial z} \left[ \frac{\partial A}{\partial z} e^{j kz} + jkAe^{j kz} \right] + k^2 Ae^{j kz} = 0
\]

\[
\nabla^2_{t} Ae^{j kz} + \frac{\partial^2 A}{\partial z^2} e^{j kz} + 2jk \frac{\partial A}{\partial z} e^{j kz} + (jk)^2 Ae^{j kz} + k^2 Ae^{j kz} = 0.
\]

Dividing by \( e^{j kz} \) and simplifying,

\[
\nabla^2_{t} A + j2k \frac{\partial}{\partial z} A + \frac{\partial^2}{\partial z^2} A = 0.
\]

The “slowly varying” approximation for \( A \) implies that:

\[
\frac{\partial^2}{\partial z^2} A \ll j2k \frac{\partial A}{\partial z}
\]

leaving,

\[
\nabla^2_{t} A + j2k \frac{\partial A}{\partial z} = 0.
\]
(b) We first evaluate a number of different derivatives:

\[
A(x, y, z) = \frac{A_1}{q} e^{jk\frac{x^2+y^2}{2q}}
\]

\[
\frac{\partial}{\partial z} A(x, y, z) = -\frac{A_1}{q^2} \frac{dq}{dz} e^{jk\frac{x^2+y^2}{2q}} - \frac{A_1}{q} \left( jk \frac{x^2+y^2}{2q^2} \right) \frac{dq}{dz} e^{jk\frac{x^2+y^2}{2q}}
\]

\[
= -\left( \frac{1}{q} + jk \frac{x^2+y^2}{2q^2} \right) \frac{dq}{dz} A(x, y, z)
\]

\[
\frac{\partial}{\partial x} A(x, y, z) = jk \frac{qA_1}{q^2} e^{jk\frac{x^2+y^2}{2q}}
\]

\[
\frac{\partial^2}{\partial x^2} A(x, y, z) = jk \frac{A_1}{q^2} e^{jk\frac{x^2+y^2}{2q}} + \left( jk \frac{x}{q} \right)^2 \frac{A_1}{q} e^{jk\frac{x^2+y^2}{2q}}
\]

\[
= \left( jk \frac{1}{q} - k^2 \frac{x^2}{q^2} \right) A(x, y, z)
\]

Similarly,

\[
\frac{\partial^2}{\partial y^2} A(x, y, z) = \left( jk \frac{1}{q} - k^2 \frac{y^2}{q^2} \right) A(x, y, z)
\]

Now substitute the partial derivatives of \( A \) into the paraxial Helmholtz equation. Noting that \( dq/dz \) is equal to 1,

\[
\nabla^2 A + j2k \frac{\partial A}{\partial z} = \left( 2jk \frac{1}{q} - k^2 \frac{x^2+y^2}{q^2} - 2jk \frac{1}{q} \frac{dq}{dz} + k^2 \frac{x^2+y^2}{q^2} \frac{dq}{dz} \right) A
\]

\[
= 0.
\]

(c) Substituting the given expression into the result from part (b),

\[
A = A_1 \left( \frac{1}{R} + j \frac{\lambda}{\pi W^2} \right) \exp \left[ jk \frac{1}{2} (x^2+y^2) \left( \frac{1}{R} + j \frac{\lambda}{\pi W^2} \right) \right]
\]

\[
= A_1 \left( \frac{1}{R} + j \frac{\lambda}{\pi W^2} \right) \exp \left[ -\frac{\rho^2}{W^2} \right] \exp \left[ jk \frac{\rho^2}{2R} \right]
\]

\[
U = A \exp[jkz]
\]

\[
= A_1 \left( \frac{1}{R} + j \frac{\lambda}{\pi W^2} \right) \exp \left[ -\frac{\rho^2}{W^2} \right] \exp \left[ jkz + jk \frac{\rho^2}{2R} \right]
\]

\[
= A_0 \frac{W_0}{W(z)} \exp \left[ -\frac{\rho^2}{W^2} \right] \exp \left[ jkz + jk \frac{\rho^2}{2R} + j\theta(z) \right]
\]

where:

\[
\frac{A_0W_0}{W} = A_1 \left[ \left( \frac{1}{R} \right)^2 + \left( \frac{\lambda}{\pi W^2} \right)^2 \right]^{1/2}
\]

\[
\theta(z) = \tan^{-1} \left( \frac{\lambda R}{\pi W^2} \right)
\]

To show that \( W_0 \) is independent of \( z \), we differentiate \( W_0^2 \) with respect to \( z \) and show
that it equals zero:

\[
W_0^2 = \left( \frac{A_1}{A_0} \right)^2 \left[ \left( \frac{W}{R} \right)^2 + \left( \frac{\lambda}{\pi W} \right)^2 \right]
\]

\[
\frac{d(W_0^2)}{dz} = \left( \frac{A_1}{A_0} \right)^2 \left( \frac{2WR^2W' - 2RW^2R'}{R^4} - \frac{2\lambda^2W'}{\pi^2W^3} \right)
\]

\[
= 2 \left( \frac{A_1}{A_0} \right)^2 \frac{\pi^2W^4RW' - \pi^2W^5R' - \lambda^2R^3W'}{\pi^2W^3R^3},
\]

where the prime sign designates a derivative with respect to z. Now, using the condition \( dq/dz = 1 \), we can express \( R' \) and \( W' \) in terms of \( R \) and \( W \):

\[
\frac{d(1/q)}{dz} = \frac{-1}{q^2} = \frac{-1}{q^2}
\]

\[
\frac{d}{dz} \left( \frac{1}{R} + j \frac{\lambda}{\pi W^2} \right) = - \left( \frac{1}{R} + j \frac{\lambda}{\pi W^2} \right)^2
\]

\[
\frac{-R'}{R^2} - 2j \frac{\lambda W'}{\pi W^3} = \frac{-1}{R^2} + \frac{\lambda^2}{\pi^2W^4} - 2j \frac{\lambda}{\pi RW^2}.
\]

Solving for both the real and the complex parts of the equation, we get

\[
R' = 1 - \frac{\lambda^2R^2}{\pi^2W^4}
\]

\[
W' = \frac{W}{R}.
\]

Substituting,

\[
\frac{d(W_0^2)}{dz} = 2 \left( \frac{A_1}{A_0} \right)^2 \frac{1}{\pi^2W^3R^3} \left[ \pi^2W^4R\frac{W}{R} - \pi^2W^5 \left( 1 - \frac{\lambda^2R^2}{\pi^2W^4} \right) - \lambda^2R^3W \right]
\]

\[
= 2 \left( \frac{A_1}{A_0} \right)^2 \frac{1}{\pi^2W^3R^3} \left( \pi^2W^5 - \pi^2W^5 + \lambda^2R^2W - \lambda^2R^2W \right)
\]

\[
= 0.
\]
Over any distance $z_k$, Fresnel propagation can be described by the transfer function

$$H(f_x, f_y; z_k) = e^{jkz_k} e^{-j\pi\lambda z_k (f_x^2 + f_y^2)}$$

Propagation over several distances $z_1, z_2, \ldots, z_n$ can be represented by multiplication of the successive transfer functions,

$$H(f_x, f_y; z_1 + z_2 + \cdots + z_n) = \prod_{k=1}^{n} H(f_x, f_y; z_k)$$

Performing the product,

$$H(f_x, f_y; z_1 + z_2 + \cdots + z_n) = e^{jk(z_1+z_2+\cdots+z_n)} e^{-j\pi\lambda(z_1+z_2+\cdots+z_n)(f_x^2+f_y^2)}$$

Clearly, since $z = z_1 + z_2 + \cdots + z_n$, propagation over distance $z$ is equivalent to propagation over the sum of the distances $z_1, z_2, \ldots, z_n$. 
4-12. (a) The Fourier transform of the amplitude transmittance function is

\[ \mathcal{F}\{t_A(\xi)\} = \sum_{n=-\infty}^{\infty} c_n \mathcal{F}\{e^{j2\pi n \xi / L}\} = \sum_{n=-\infty}^{\infty} c_n \delta\left(f_x - \frac{n}{L}\right). \]

Assuming unit amplitude, normally incident plane wave illumination, the intensity in any order will be proportional to the squared magnitude of the Fourier coefficient associated with that order. More generally, for arbitrary strength of illumination, the diffraction efficiency of a given order is the squared magnitude of the Fourier coefficient of the delta function corresponding to that order. Thus

\[ \eta_k = |c_k|^2. \]

(b) We must find the Fourier coefficients of the amplitude transmittance function

\[ t_A(\xi) = \left| \cos\left(\frac{\pi \xi}{L}\right) \right|. \]

Do so as follows:

\[ c_k = \frac{1}{L} \int_{-L/2}^{L/2} \left| \cos\left(\frac{\pi \xi}{L}\right) \right| e^{-j \frac{2\pi k \xi}{L}} d\xi = \frac{1}{L} \mathcal{F}\left\{ \text{rect}\left(\frac{\xi}{L}\right) \cos\left(\frac{\pi \xi}{L}\right) \right\}_{f_x=k/L} \]

\[ = \frac{1}{2L} \left( L \text{sinc}\left[ L\left(f_x - \frac{1}{2L}\right) \right] + L \text{sinc}\left[ L(f_x + \frac{1}{2L}) \right] \right)_{f_x=k/L} \]

\[ = \frac{1}{2} \left[ \text{sinc}\left( \frac{2k - 1}{2} \right) + \text{sinc}\left( \frac{2k + 1}{2} \right) \right]. \]

The diffraction efficiency is seen to be

\[ \eta_k = |c_k|^2 = \frac{1}{4} \left[ \text{sinc}\left( \frac{2k - 1}{2} \right) + \text{sinc}\left( \frac{2k + 1}{2} \right) \right]^2. \]
For the particular case of the first diffraction order \((k = 1)\),

\[
|c_1|^2 = \frac{1}{4} \left[ \text{sinc} \left( \frac{1}{2} \right) + \text{sinc} \left( \frac{3}{2} \right) \right]^2 = \frac{1}{4} \left[ \frac{2}{\pi} - \frac{2}{3\pi} \right]^2 = \frac{4}{9\pi^2} = 4.5\%.
\]
4-13. We first obtain a Fourier series expansion of the grating amplitude transmittance, and then use the results of the previous problem. First note that in the region $|\xi| \leq L/2$, the amplitude transmittance can be written

$$t_A(\xi) = \left(\frac{1}{2} - t_m\right) \text{rect} \left(\frac{\xi}{L}\right) + 2t_m \text{rect} \left(\frac{\xi}{L/2}\right).$$

Thus

$$c_n = \frac{1}{L} \mathcal{F} \left\{ \left(\frac{1}{2} - t_m\right) \text{rect} \left(\frac{\xi}{L}\right) + 2t_m \text{rect} \left(\frac{\xi}{L/2}\right) \right\}_{f_X = n/L}$$

$$= \left(\frac{1}{2} - t_m\right) \text{sinc}(n) + t_m \text{sinc} \left(\frac{n}{2}\right).$$

(a) The fraction of light absorbed by the grating is found by subtracting the spatial average (i.e. an average over one period) of $|t_X(\xi)|^2$ from unity,

$$\text{fraction absorbed} = 1 - \frac{1}{L} \int_{-L/2}^{L/2} |t_A(\xi)|^2 d\xi$$

$$= 1 - \frac{1}{L} \left[ \frac{L}{2} \left(\frac{1}{2} - t_m\right)^2 + \frac{L}{2} \left(\frac{1}{2} + t_m\right)^2 \right] = \frac{3}{4} - t_m^2.$$

(b) The fraction of light transmitted by the grating is simply 1 minus the fraction absorbed. Therefore

$$\text{fraction transmitted} = \frac{1}{4} + t_m^2.$$

(c) The fraction of light appearing in a single first order will be given by $|c_1|^2$. We have

$$|c_1|^2 = \left(\frac{2t_m}{\pi}\right)^2 = \frac{4t_m^2}{\pi^2}.$$