1. a) General H-F integral for broadband light from class:

$$u(Q,t) = \iint_{\Sigma} \frac{\cos(\hat{n}, \vec{r}_{01})}{2\pi c r_{01}} \frac{d}{dt} u(P_1, t - \frac{r_{01}}{c}) \, ds$$

for small pinholes, the field from each pinhole is just:

$$u(Q,t) = \frac{A_1 \cos(\hat{n}, \vec{r}_{01})}{2\pi c r_{01}} \frac{d}{dt} u(P_1, t - \frac{r_{01}}{c})$$

the field from 2 pinholes is then

$$u_{TOT}(Q,t) = K_1 \frac{d}{dt} u(P_1, t - \frac{r_{01}}{c}) + K_2 \frac{d}{dt} u(P_2, t - \frac{r_{02}}{c})$$

b) Taking $I = \langle |u_{TOT}(Q,t)|^2 \rangle$

$$I(Q) = I^{(1)}(Q) + I^{(2)}(Q) + K_1 K_2 \langle \frac{d}{dt} u(P_1, t - \frac{r_{01}}{c}) \frac{d}{dt} u^*(P_2, t - \frac{r_{02}}{c}) \rangle$$
$$+ K_1 K_2 \langle \frac{d}{dt} u^*(P_1, t - \frac{r_{01}}{c}) \frac{d}{dt} u(P_2, t - \frac{r_{02}}{c}) \rangle$$

(K_1, K_2 are both real)

consider the expression:

$$\langle \frac{d}{dt} u(P_1, t+\tau) \frac{d}{dt} u^*(P_2, t) \rangle$$

The definition of the time average tells us this is written:

$$\lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \frac{d}{dt} u(P_1, t+\tau) \frac{d}{dt} u^*(P_2, t) \, dt$$

integrate by parts

$$\lim_{T \to \infty} \frac{1}{T} \left\{ \frac{d}{dt} u(P_1, t+\tau) u^*(P_2, t) \Big|_{-\frac{T}{2}}^{\frac{T}{2}} - \int_{-T/2}^{T/2} \frac{d^2}{dt^2} u(P_1, t+\tau) u^*(P_2, t) \right\}$$

The first term goes to zero in the limit $T \to \infty$, so finally we have shown that

$$\left\langle \frac{d}{dt} u(P_1, t+\tau) \frac{d}{dt} u^*(P_2, t) \right\rangle = -\left\langle \frac{d^2}{dt^2} u(P_1, t+\tau) u^*(P_2, t) \right\rangle$$

but it is also true by definition of the derivative that

$$\frac{d^2}{dt^2} u(P_1, t+\tau) = \frac{\partial^2}{\partial \tau^2} u(P_1, t+\tau)$$

Thus $\left\langle \frac{d}{dt} u(P_1, t+\tau) \frac{d}{dt} u^*(P_2, t) \right\rangle = -\frac{\partial^2}{\partial \tau^2} \langle u(P_1, t+\tau) u^*(P_2, t) \rangle$.

Therefore

$$I(Q) = I^{(1)}(Q) + I^{(2)}(Q) - K_1 K_2 \frac{\partial^2}{\partial \tau^2} \langle u(P_1, t+\tau) u^*(P_2, t) \rangle$$

$$- K_1 K_2 \frac{\partial^2}{\partial \tau^2} \langle u^*(P_1, t+\tau) u(P_2, t) \rangle$$

$$= I^{(1)}(Q) + I^{(2)}(Q) - K_1 K_2 \left[\frac{\partial^2}{\partial \tau^2} \Gamma_{12}(\tau) + \frac{\partial^2}{\partial \tau^2} \Gamma_{12}^*(\tau) \right]$$

$$= I^{(1)}(Q) + I^{(2)}(Q) - 2 K_1 K_2 \, \mathrm{Re} \left[\frac{\partial^2}{\partial \tau^2} \Gamma_{12}(\tau) \right]$$

c) we can write, in general, the inverse transform:

$$u(P_1, t+\tau) = \int_{-\infty}^{\infty} U(P_1, \nu) \exp(j 2\pi \nu [t+\tau]) d\nu$$

substitute this into:

$$\frac{\partial^2}{\partial \tau^2} \langle u(P_1, t+\tau) u^*(P_2, t) \rangle = \frac{\partial^2}{\partial \tau^2} \langle \int_{-\infty}^{\infty} U(P_1, \nu) e^{j2\pi\nu(t+\tau)} d\nu \; u^*(P_2, t) \rangle$$

$$= -4\pi^2 \langle \int_{-\infty}^{\infty} \nu^2 U(P_1, \nu) e^{j2\pi\nu(t+\tau)} d\nu \; u^*(P_2, t) \rangle$$

if u_1 is narrowband, this reduces to:

$$-4\pi^2 \nu^2 \langle u(P_1, t+\tau) u^*(P_2, t) \rangle$$

$$= -4\pi^2 \nu^2 \Gamma_{12}(\tau)$$

therefore, for narrowband light

$$I(Q) = I^{(1)}(Q) + I^{(2)}(Q) + 2(4\pi^2\nu^2) K_1 K_2 \, Re \, \Gamma_{12}(\tau)$$

our previous narrowband case result was

$$I^{(1)}(Q) + I^{(2)}(Q) + 2 \mathcal{K}_1 \mathcal{K}_2 Re \, \Gamma_{12}(\tau)$$

but our $K_1 = \dfrac{A_1 \cos(\hat{n}, \vec{r}_1)}{2\pi c r_1}$

and $\mathcal{K}_1 = \dfrac{A_1 \cos(\hat{n}, \vec{r}_1)}{\lambda r'}$. So $4\pi^2\nu^2 K_1 K_2 = \mathcal{K}_1 \mathcal{K}_2$

by similar reasoning $I^{(1)}(Q)$ and $I^{(2)}(Q)$ for broadband light reduce to the narrowband expressions.

2. <u>Image Intensity</u> : $I_i(u,v)$

mutual intensity: $J_i(u_1, v_1; u_2, v_2)$

$$I_i(u,v) = J_i(u,v;u,v) \quad (A)$$

Fourier transform of mutual intensity: $\mathcal{J}_i(\nu_1, \nu_2, \nu_3, \nu_4)$

$$J_i(u_1;v_1;u_2,v_2) = \mathcal{F}^{-1}[\mathcal{J}_i(\nu_1,\nu_2,\nu_3,\nu_4)]$$

$$= \iiiint_{-\infty}^{\infty} \mathcal{J}_i(\nu_1,\nu_2,\nu_3,\nu_4)\exp[-j2\pi(\nu_1 u_1 + \nu_2 v_1 + \nu_3 u_2 + \nu_4 v_2)]\,d\nu_1 d\nu_2 d\nu_3 d\nu_4$$

take $u_1 = u_2 = u$ $\quad v_1 = v_2 = v$ to get $I(u,v)$, using (A)

$$I(u,v) = \iiiint_{-\infty}^{\infty} \mathcal{J}_i(\nu_1,\nu_2,\nu_3,\nu_4)\exp\{-j2\pi[(\nu_1+\nu_3)u + (\nu_2+\nu_4)v]\}\,d\nu_1 d\nu_2 d\nu_3 d\nu_4$$

3. <u>Image Spectrum</u>

by definition: $\mathcal{I}_i(\nu_u, \nu_v) = \iint_{-\infty}^{\infty} I(u,v)\exp[j2\pi(u\nu_u + v\nu_v)]\,du\,dv$

using previous problem 2, we have:

$$\mathcal{I}_i(\nu_u,\nu_v) = \iiiint\!\!\iint_{-\infty}^{\infty} \mathcal{J}_i(\nu_1,\nu_2,\nu_3,\nu_4)\exp\{j2\pi[(\nu_u-\nu_1-\nu_3)u + (\nu_v-\nu_2-\nu_4)v]\}\,d\nu_1 d\nu_2 d\nu_3 d\nu_4\,du\,dv$$

changing order of integration

$$= \iiiint_{-\infty}^{\infty} d\nu_1 d\nu_2 d\nu_3 d\nu_4\, \mathcal{J}_i(\nu_1,\nu_2,\nu_3,\nu_4)\underbrace{\iint_{-\infty}^{\infty} du\,dv\,\exp\{j2\pi[(\nu_u-\nu_1-\nu_3)u + (\nu_v-\nu_2-\nu_4)v]\}}_{\delta(\nu_u-\nu_1-\nu_3)\,\delta(\nu_v-\nu_2-\nu_4)}$$

perform the integrals over ν_3, ν_4. The δ functions sift out giving:

$$\mathcal{I}_i(\nu_u,\nu_v) = \iint_{-\infty}^{\infty} d\nu_1 d\nu_2\, \mathcal{J}_i(\nu_1,\nu_2,\nu_u-\nu_1,\nu_v-\nu_2) \quad \checkmark$$

compute the complex coherence factor in the object plane. For the telecentric system, following the hint,

$$\mu(\Delta\xi, \Delta\eta) = J_0\left(\frac{2\pi}{\bar{\lambda} f}\rho\sqrt{\Delta\xi^2 + \Delta\eta^2}\right)$$

where ρ = radius of the annular source, $\Delta\xi$ and $\Delta\eta$ are separations in the object plane. For a pinhole separation of $X = 1.22\,\bar{\lambda} f/D$,

$$\mu(\Delta\xi = X, \Delta\eta = 0) = J_0\left(2\pi\rho \cdot \frac{1.22}{D}\right)$$

(a) The smallest annulus radius for which the pinholes are illuminated incoherently ($\mu = 0$) occurs when the argument of the Bessel function is ~ 2.405 (i.e. the first zero of the Bessel function).

$$\therefore \quad 2\pi\rho \cdot \frac{1.22}{D} \simeq 2.405 \qquad \boxed{\rho_{incoh} = 0.314D}$$

(b) For the central value to achieve the "largest dip", we find the argument that causes the Bessel function to take on its most negative value. This occurs for

$$2\pi\rho \cdot \frac{1.22}{D} \simeq 3.83$$

$$\boxed{\rho_{\substack{Big\\Dip}} = 0.50\,D}$$

For a given object intensity distribution $I(x)$, we know (apart from a scaling constant) that $\mu_{12} = \mathcal{F}\{I(x)\}$. Since we are given $|\mu_{12}|$, we can calculate the autocorrelation of $I(x)$ using

$$I(x) \star I(x) = \mathcal{F}^{-1}\{|\mu_{12}|^2\}.$$

Thus, the problem can be restated: Given a 1-D symmetric object, can we determine $I(x)$ from its autocorrelation $I(x) \star I(x)$?

Let $I(x)$ be represented by N samples:

$$I(x_1),\ I(x_2),\ \ldots\ I(x_N)$$

(Note: Often the data string is padded with zeros — this is <u>not</u> done here.) Denoting $I_n = I(x_n)$, we can write the autocorrelation of I_n as

$$A_K = I_n \star I_n = \sum_{n=1}^{N} I_n I_{n+K}$$

where

$$K = -N,\ -N+1,\ \ldots,\ N-1,\ N$$

and

$$I_n = 0 \quad \text{for} \quad n > N,\ n < 1.$$

Since the object is symmetric,

$$I_n = I_{N-n+1}.$$

So,

$$A_N = 0$$

$$A_{N-1} = I_1 I_N = I_1^2 \Rightarrow I_1 = \sqrt{A_{N-1}}$$

$$A_{N-2} = I_1 I_{N-1} + I_2 I_N$$

$$= 2 I_1 I_2 \Rightarrow I_2 = \frac{A_{N-2}}{2 I_1}$$

etc...

In this way we uniquely determine the object intensity samples $I_1, I_2, \ldots I_N$.

6. Statistical Optics 7-10

(a) The telecentric optics (Fig 7-11) act to image the source in the pupil plane (x, y - plane), where the object transparency acts as the pupil function generating a shift-invariant imaging kernel

$$\underline{\tilde{K}}(x-\alpha, y-\beta) = \mathcal{F}\{\underline{\tilde{t}_o}(\xi, \eta)\}\Big|_{\xi = x-\alpha \quad \eta = \alpha}$$

Prob 7-10 (Continued):

So

$$\tilde{\underline{K}}(x-\alpha, y-\beta) = \mathcal{F}\{\cos 2\pi \nu_0 \xi\}\Big|_{\frac{x-\alpha}{\lambda f},\ \frac{y-\beta}{\lambda f}}$$

$$= \frac{1}{2}\delta\left(\frac{x-\alpha}{\lambda f} - \nu_0\right)\delta\left(\frac{y-\beta}{\lambda f}\right)$$

$$+ \frac{1}{2}\delta\left(\frac{x-\alpha}{\lambda f} + \nu_0\right)\delta\left(\frac{y-\beta}{\lambda f}\right)$$

$$= \frac{(\lambda f)^2}{2}\Big[\delta(x-\alpha-\lambda f \nu_0)\cdot\delta(y-\beta)$$

$$+ \delta(x-\alpha+\lambda f \nu_0)\cdot\delta(y-\beta)\Big]$$

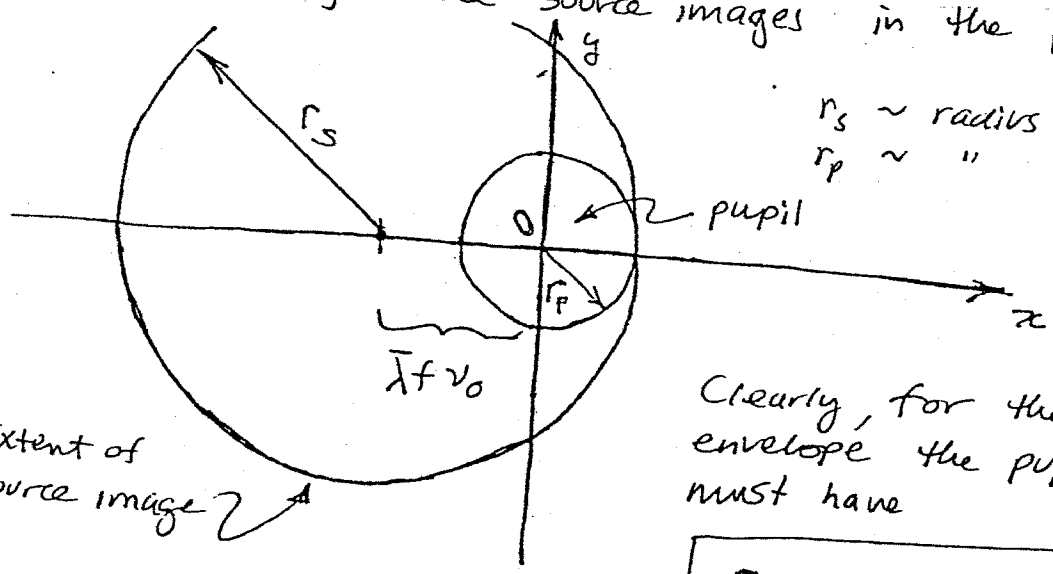
and so, the pupil field phasor is

$$\underline{A}_p(x,y) = \iint_{-\infty}^{\infty} \tilde{\underline{K}}(x-\alpha, y-\beta)\, \underbrace{\underline{A}_s(\alpha,\beta)}_{\text{phasor amplitude of the source}}\, d\alpha\, d\beta$$

$$= \frac{(\lambda f)^2}{2}\Big[\underline{A}_s(x-\lambda f \nu_0) + \underline{A}_s(x+\lambda f \nu_0)\Big]$$

which is the sum of 2 replications of the source located at $x = \pm \lambda f \nu_0$, $y = 0$.

(b) Sketching these source images in the pupil plane,



r_s ~ radius of source
r_p ~ " " pupil

Clearly, for the source to envelope the pupil, we must have

$$\boxed{r_s \geq r_p + \lambda f \nu_0}$$

7. Fourier Optics 9-1

9-1. A complicated but correct solution to the problem would write all the fields incident on the film, find the intensity, and find the fields transmitted by the hologram. A much simpler solution is based on Eq. (9-38) with $\lambda_2 = \lambda_1$. That equation states that

$$\frac{1}{z_i} = \frac{1}{z_p} \pm \frac{1}{z_r} \mp \frac{1}{z_o}.$$

This equation should now be compared with the lens law, which we must adapt to the sign convention used in the discussion of holographic image locations. Remembering that z_o is negative for an object to the left of the hologram or lens, the lens law can be written

$$\frac{1}{z_i} = \frac{1}{f} + \frac{1}{z_o}.$$

Equating these two expressions for $1/z_i$, we see immediately that

$$\frac{1}{f} = \frac{1}{z_p} \pm \frac{1}{z_r} \mp \frac{1}{z_o} - \frac{1}{z_o},$$

yielding two focal lengths

$$f = \left(\frac{1}{z_p} - \frac{1}{z_r}\right)^{-1} \quad \text{and} \quad f = \left(\frac{1}{z_p} + \frac{1}{z_r} - \frac{2}{z_o}\right)^{-1}.$$

Take first the case of $z_p = z_r$. The image distance is

$$z_i = \left(\frac{1}{z_r} \pm \frac{1}{z_r} \mp \frac{1}{z_o}\right)^{-1}.$$

Thus the two solutions are (lower set of signs first)

$$z_i = z_o \qquad \text{and} \qquad z_i = \left(\frac{2}{z_r} - \frac{1}{z_o}\right)^{-1} = \frac{z_r z_o}{2z_o - z_r}.$$

Note that since $z_o < 0$ (i.e. the object lies to the left of the hologram), the first image also lies to the left of the hologram (a virtual image), while the second can lie to the left or to the right, depending on the particular values of z_o and z_r. The transverse magnifications in the two cases are

$$M_t = 1 \qquad \text{and} \qquad M_t = \left|\frac{z_i}{z_o}\right| = \left|\frac{z_r}{2z_o - z_r}\right|.$$

Thus one of the images is virtual and has unit transverse magnification.

Now consider the case for $z_p = -z_r$. The two solutions for image distance are now (again lower set of signs first)

$$z_i = \left(-\frac{2}{z_r} + \frac{1}{z_o}\right)^{-1} = \frac{-z_r z_o}{2z_o - z_r} \qquad \text{and} \qquad z_i = -z_o.$$

The transverse magnifications in the two cases are

$$M_t = \left|\frac{z_i}{z_o}\right| = \left|\frac{z_r}{2z_o - z_r}\right| \qquad \text{and} \qquad M_t = 1.$$

Note that, since z_o is negative, the second image is real (z_i positive) and has unit transverse magnification.

9. Fourier Optics 9-5

9-5. (a) We know the magnification to be 2 from the object and image sizes given. We also know

$$M_t = \left|\frac{\lambda_2 z_i}{\lambda_1 z_o}\right| = 2.$$

Choosing the appropriate signs by noting that $z_i > 0$ and $z_o < 0$,

$$z_o = -\frac{\lambda_2 z_i}{\lambda_1 \times 2} = -\frac{(488\,\text{nm})(1\,\text{m})}{(632.8\,\text{nm})(2)} = -0.38\,\text{m}.$$

Using this result in the expression for z_i,

$$z_i = \left(\frac{1}{z_p} \pm \frac{\lambda_2}{\lambda_1 z_r} \mp \frac{\lambda_2}{\lambda_1 z_o}\right)^{-1} = \left(\frac{1}{z_p} \pm \frac{\lambda_2}{\lambda_1 z_r} \pm \frac{2}{z_i}\right)^{-1}.$$

Solving for z_r,

$$\begin{aligned} z_r &= \frac{\lambda_2}{\lambda_1}\left(-\frac{2}{z_i} \pm \frac{1}{z_i} \mp \frac{1}{z_p}\right)^{-1} \\ &= -\frac{\lambda_2}{\lambda_1}\left(\frac{-1}{z_i} - \frac{1}{z_p}\right)^{-1} \quad \text{or} \quad \frac{\lambda_2}{\lambda_1}\left(\frac{-3}{z_i} + \frac{1}{z_p}\right)^{-1} \\ &= 0.77\,\text{m} \quad \text{or} \quad -0.15\,\text{m}. \end{aligned}$$

Since the problem specifies that $z_r < 0$,

$$z_r = -0.15\,\text{m}.$$

(b) The same effect as flipping the hologram can be obtained by keeping the hologram unflipped but exposing the film from the other side. This has the effect of reversing the signs of z_o and z_r, making both of them positive. z_o must now be $+0.38$ m to assure $M_t = 2$. Carrying out the same calculations as part (a) with the new signs for z_o and z_r, we find

$$\begin{aligned} z_o &= +0.38\,\text{m} \\ z_r &= +0.15\,\text{m}. \end{aligned}$$

10. Fourier Optics 9-6

9-6. (a) We can find the maximum spatial frequency with the help of the figure. The maximum

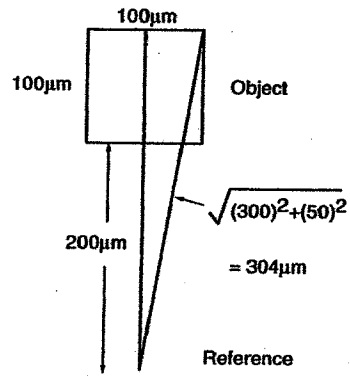


Figure 9-15: .

spatial frequency will be the maximum distance from the reference point to any point on the object (304 μm in this case), divided by λz_o,

$$f\text{max} = \frac{l}{\lambda z_o} = \frac{304\,\mu\text{m}}{1 \times 10^{-4}\,\mu\text{m} \times 20\,\text{mm}} = 152000 \text{ cycles/mm}.$$

(b) The experiment will fail because the periods of all components of the holographic grating are much smaller than the wavelength of the reconstruction source. As a consequence, all diffraction orders will be evanescent, and there will be no way to form an image.

9-11. First calculate the bandwidth of the object. Since the Fourier-transform hologram dimensions are (from the discussion of Section 9.9.1) $L_X \times L_Y$, the bandwidths of interest are

$$\begin{aligned} 2b_X &= \frac{L_X}{\lambda f} \\ 2b_Y &= \frac{L_Y}{\lambda f}. \end{aligned}$$

Thus given an object of dimensions $L_\xi \times L_\eta$, if the object is sampled at the Nyquist rate, the number of samples will be

$$\begin{aligned} n_X &= L_\xi \times 2b_X = \frac{L_\xi L_X}{\lambda f} \\ n_Y &= L_\eta \times 2b_Y = \frac{L_\eta L_Y}{\lambda f}. \end{aligned}$$

This is precisely the same number of samples required in the Fourier plane, as evidenced by Eq. (9-90).

12. Fourier Optics 9-14

9-14. For simplicity, assume that the reference point is on the optical axis. The spatial frequency associated with the fringe pattern generated by interference of this reference with a point-source object distance d away from the reference point will be

$$f_o = \frac{d}{\lambda z}.$$

Equating f_o to the cutoff frequency f_c of each type of film and solving for the resulting value of d, we obtain:

$$d = \lambda z f_c = \begin{cases} \text{Tri-X} & 1.6\,\text{mm} \\ \text{High-Contrast Copy} & 1.9\,\text{mm} \\ \text{SO-243} & 9.5\,\text{mm} \\ \text{Agepan FF} & 19.0\,\text{mm} \end{cases}$$