EE C245 – ME C218
Introduction to MEMS Design
Fall 2007

Prof. Clark T.-C. Nguyen

Dept. of Electrical Engineering & Computer Sciences
University of California at Berkeley
Berkeley, CA 94720

Lecture 15: Beam Combos

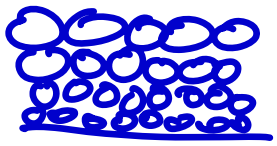
Lecture Outline

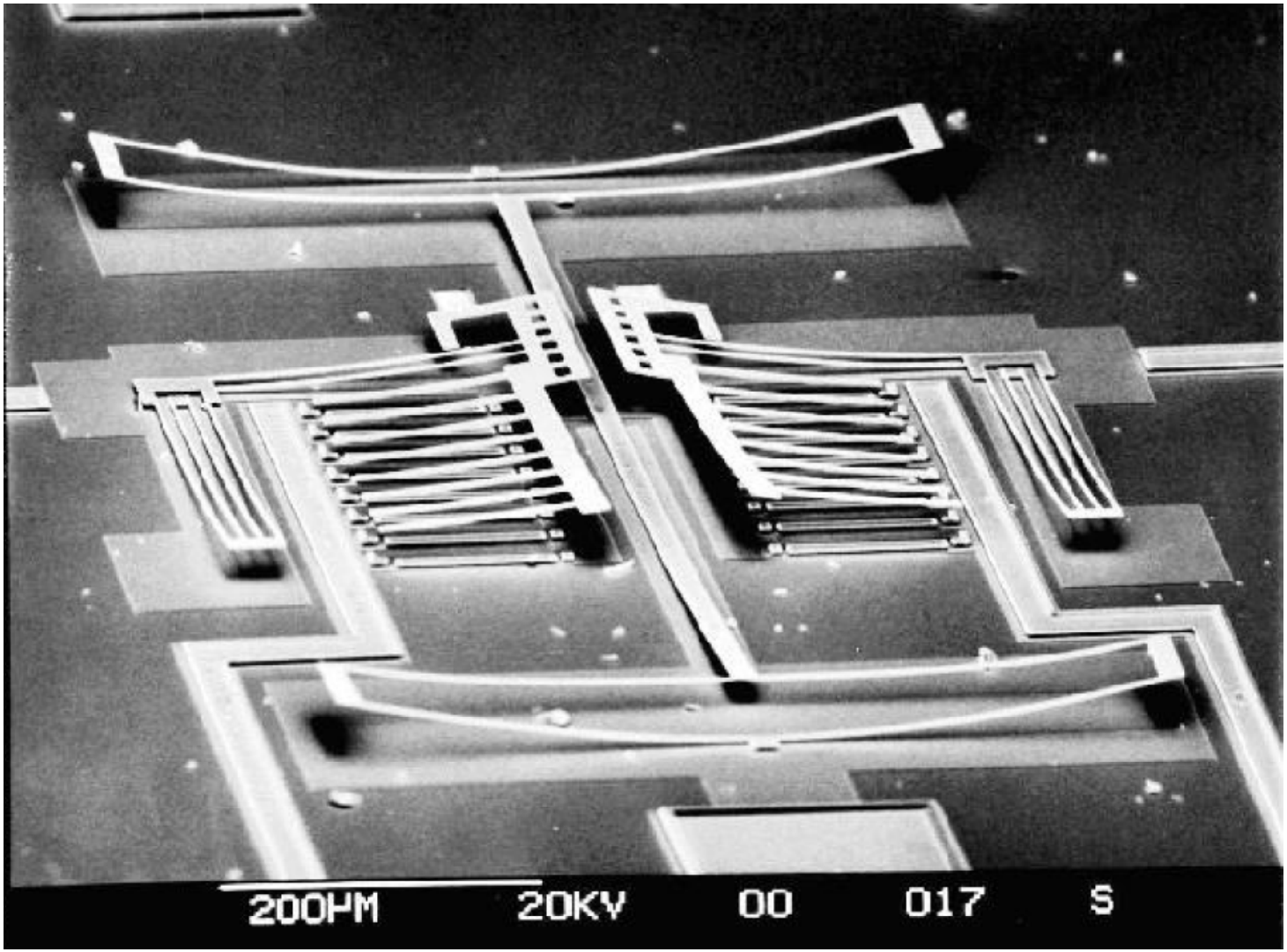
- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients

Stress Gradients in Cantilevers

Vertical Stress Gradients

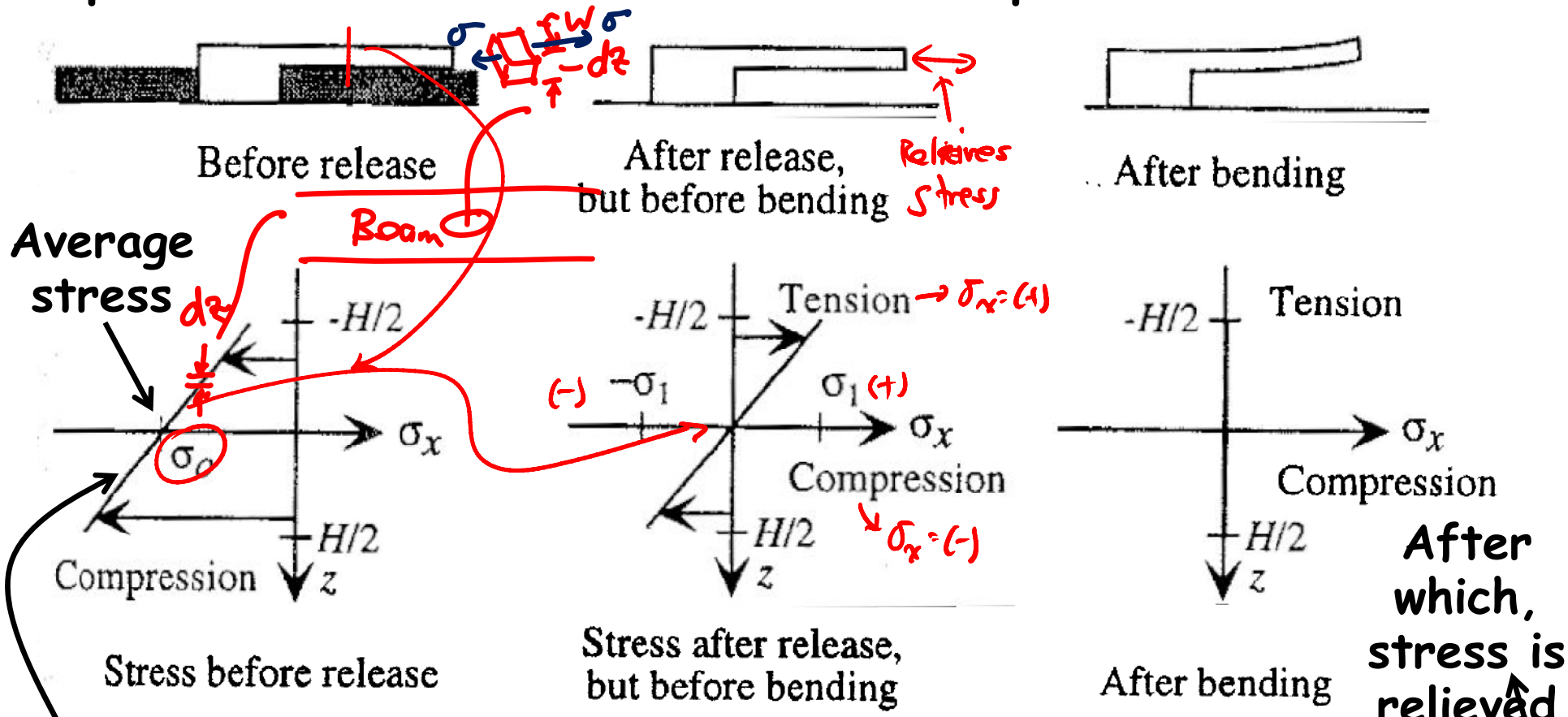
- Variation of residual stress in the direction of film growth
- Can warp released structures in z-direction





Stress Gradients in Cantilevers

- **Below**: surface micromachined cantilever deposited at a high temperature then cooled → assume compressive stress



Stress gradient

Once released, beam length increases slightly to relieve average stress

But stress gradient remains → induces moment that bends beam

After which, stress is relieved

Stress Gradients in Cantilevers (cont)

Find the radius of curvature.

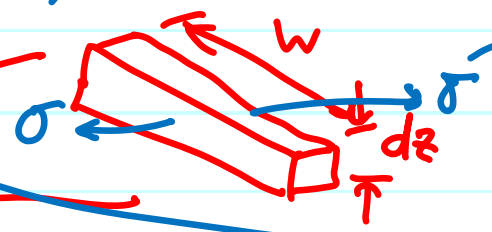
Prior to release, axial stress is: $\sigma = \sigma_0 - \frac{\sigma_1}{(H/2)} z$ (for the leftmost curve of the previous page)

Find the internal bending moment:

$$M_x = \int_{-H/2}^{H/2} \left[(W \cdot dz)\sigma \right] \cdot z$$

width

$= f(z)$

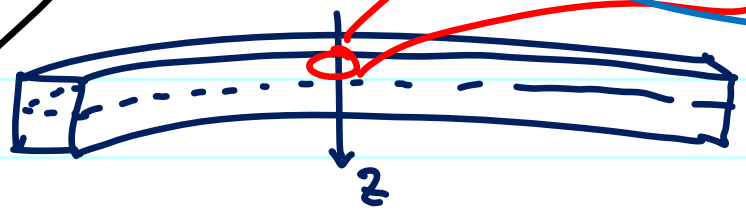


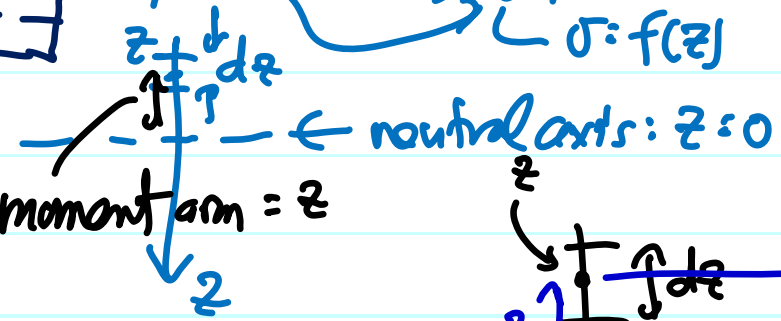
$\frac{F}{A}$

$A = W dz$

$\sigma \cdot A = F$

$\sigma = f(z)$



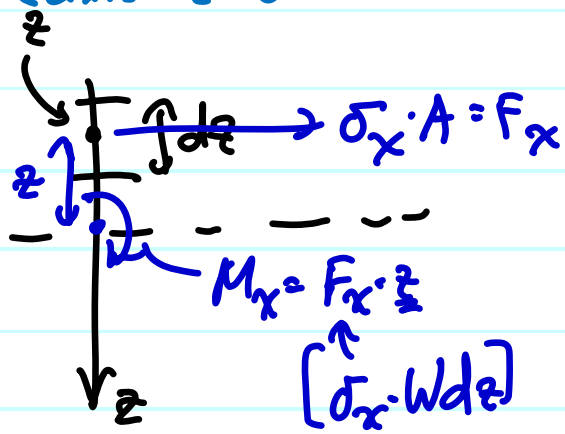


$$= W \int_{-H/2}^{H/2} \left(z\sigma_0 - \frac{\sigma_1 z^2}{(H/2)} \right) dz$$

$$= W \left(\frac{1}{2}\sigma_0 z^2 - \frac{2\sigma_1 z^3}{3H} \right) \Big|_{-H/2}^{H/2}$$

$$= W \left(\cancel{\frac{1}{2}\sigma_0 \frac{H^2}{4}} - \frac{2}{3}\sigma_1 \frac{H^2}{8} - \cancel{\frac{1}{2}\sigma_0 \frac{H^2}{4}} - \frac{2\sigma_1 H^2}{3(8)} \right)$$

$$= \boxed{-\frac{1}{6}\sigma_1 W H^2 = M_x}$$



Radius of Curvature f/ Stress Gradient

Thus, the radius of curvature is:

$$\frac{1}{R} = -\frac{M_x}{E'I} \rightarrow R = -\frac{E'I}{M_x} = -\frac{E'\left(\frac{1}{12}WH^3\right)}{-\frac{1}{6}\sigma_1 W H^2} = \frac{1}{2}\frac{E'H}{\sigma_1}$$

Biaxial Modulus

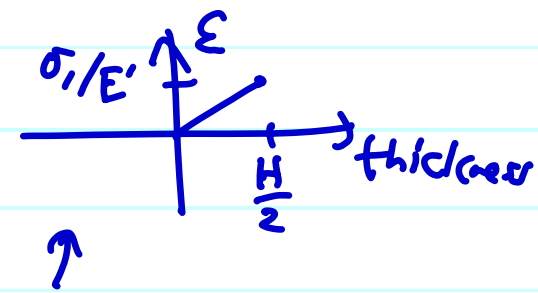
$$\left[I = \frac{1}{12}Wh^3\right]$$

$$\Rightarrow \boxed{R = \frac{1}{2}\frac{E}{(1-\nu)}\frac{H}{\sigma_1}}$$

Radius of Curvature for a Cantilever w/ Stress Gradient

$$\boxed{\sigma_1 = \frac{1}{2}\frac{E}{(1-\nu)}\frac{H}{R}}$$

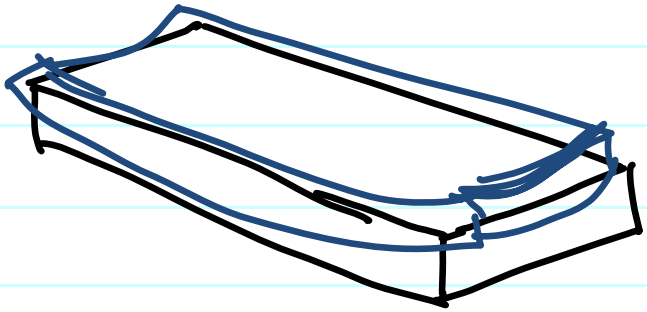
$\Rightarrow$ R can be used to determine stress gradient



Definition.

Strain Gradient, Γ = slope of the strain-thickness curve

$$\Gamma = \frac{\sigma_1/E'}{H/2} = \frac{1}{2}\frac{\sigma_1}{H}\frac{1}{E'}$$

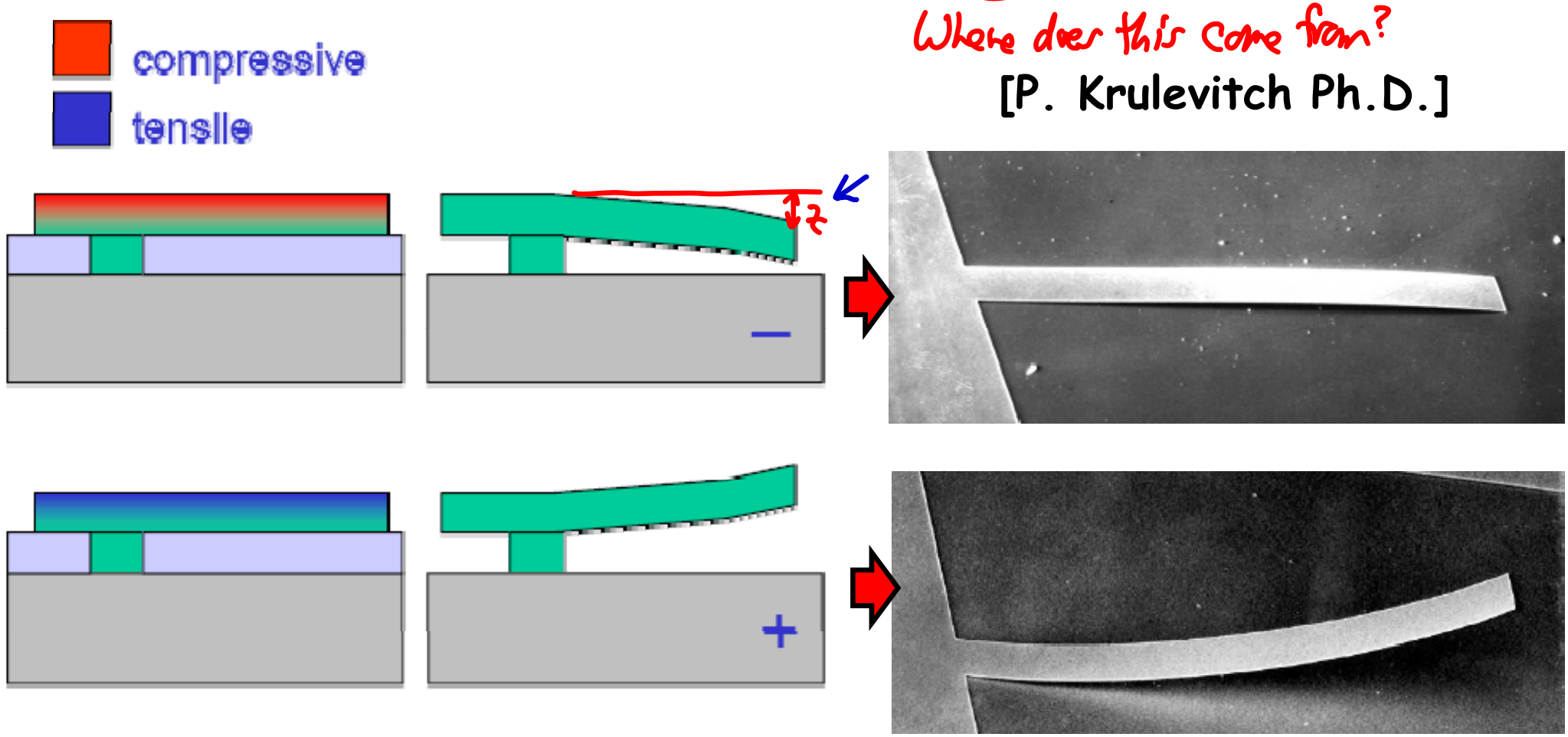


Measurement of Stress Gradient

- Use cantilever beams
  - Strain gradient (Γ = slope of stress-thickness curve) causes beams to deflect up or down
  - Assuming linear strain gradient Γ, $z = \Gamma L^2/2$

Where does this come from?

[P. Krulevitch Ph.D.]



Tip Bending Distance

To find the amount of bending @ the tip of the cantilever:

→ go back to:

$$\underbrace{E' I_z}\frac{d^2w}{dx^2} = -M_x = \underbrace{\frac{1}{6}\sigma_1 W H^2}$$

$$\left[I_z = \frac{1}{12}WH^3\right]$$

$$\Gamma = \frac{\sigma_1/E'}{H/2} \rightarrow \sigma_1 = \Gamma E'\left(\frac{H}{2}\right)$$

$$\rightarrow \cancel{E'\left(\frac{1}{12}WH^3\right)}\frac{d^2w}{dx^2} = \frac{1}{6}\Gamma E'\left(\frac{H}{2}\right)WH^2$$

$$= \cancel{E'\left(\frac{1}{12}WH^3\right)}\Gamma$$

$$\therefore \boxed{\frac{d^2w}{dx^2} = \Gamma}$$

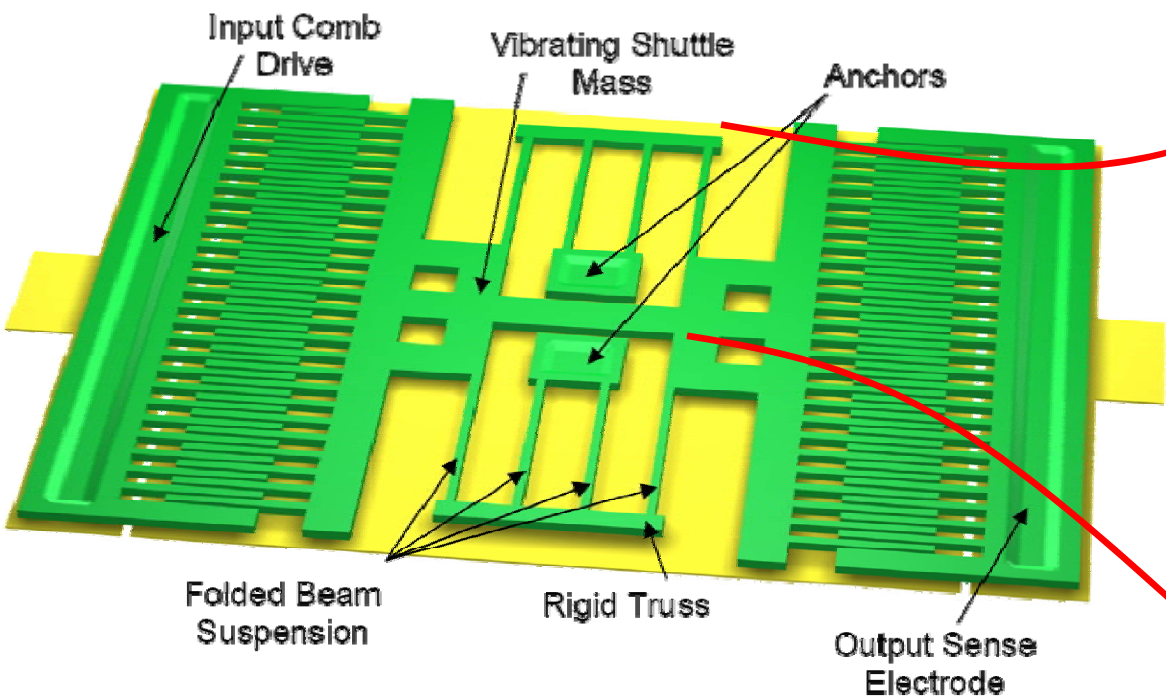
↓ Integrate 2x

$$w = \frac{1}{2}\Gamma x^2\Big|_0^L = \boxed{\frac{1}{2}\Gamma L^2 = \text{Tip Displacement}} \checkmark$$

Folded-Flexure Suspensions

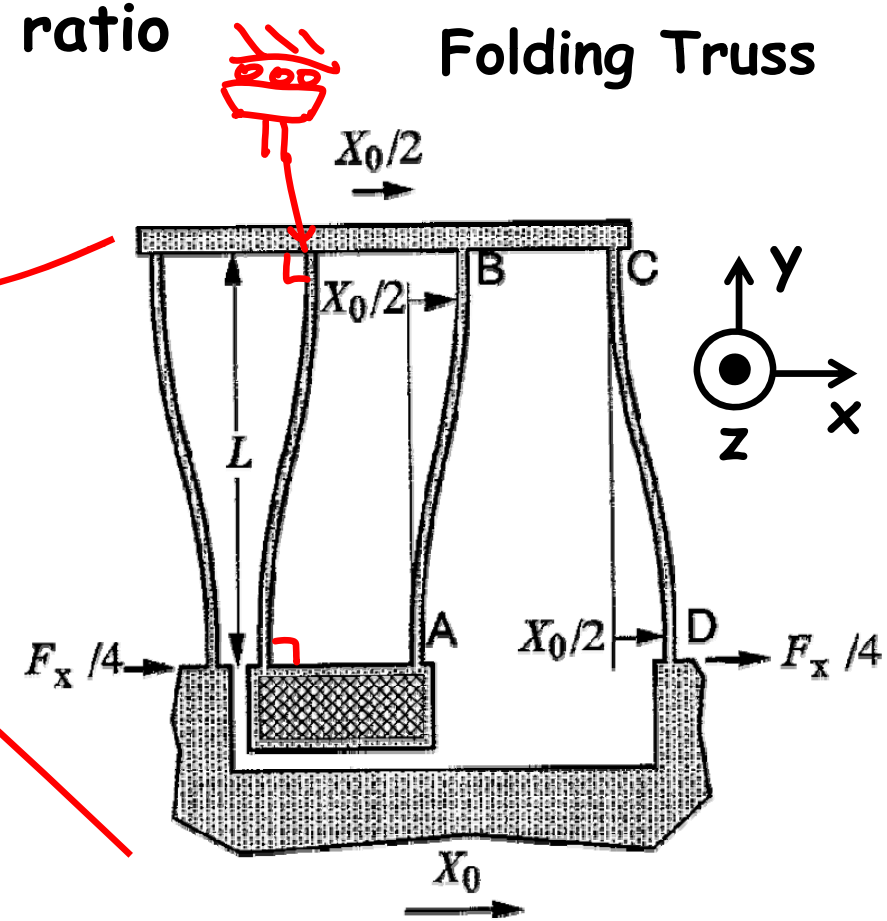
Folded-Beam Suspension

- Use of folded-beam suspension brings many benefits
  - Stress relief: folding truss is free to move in y-direction, so beams can expand and contract more readily to relieve stress
  - High y-axis to x-axis stiffness ratio



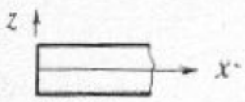
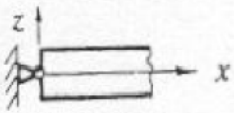
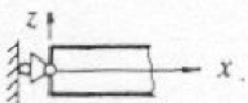
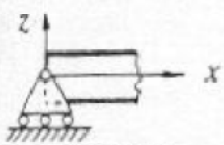
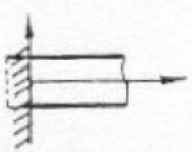
Comb-Driven Folded Beam Actuator

Folding Truss



Beam End Conditions

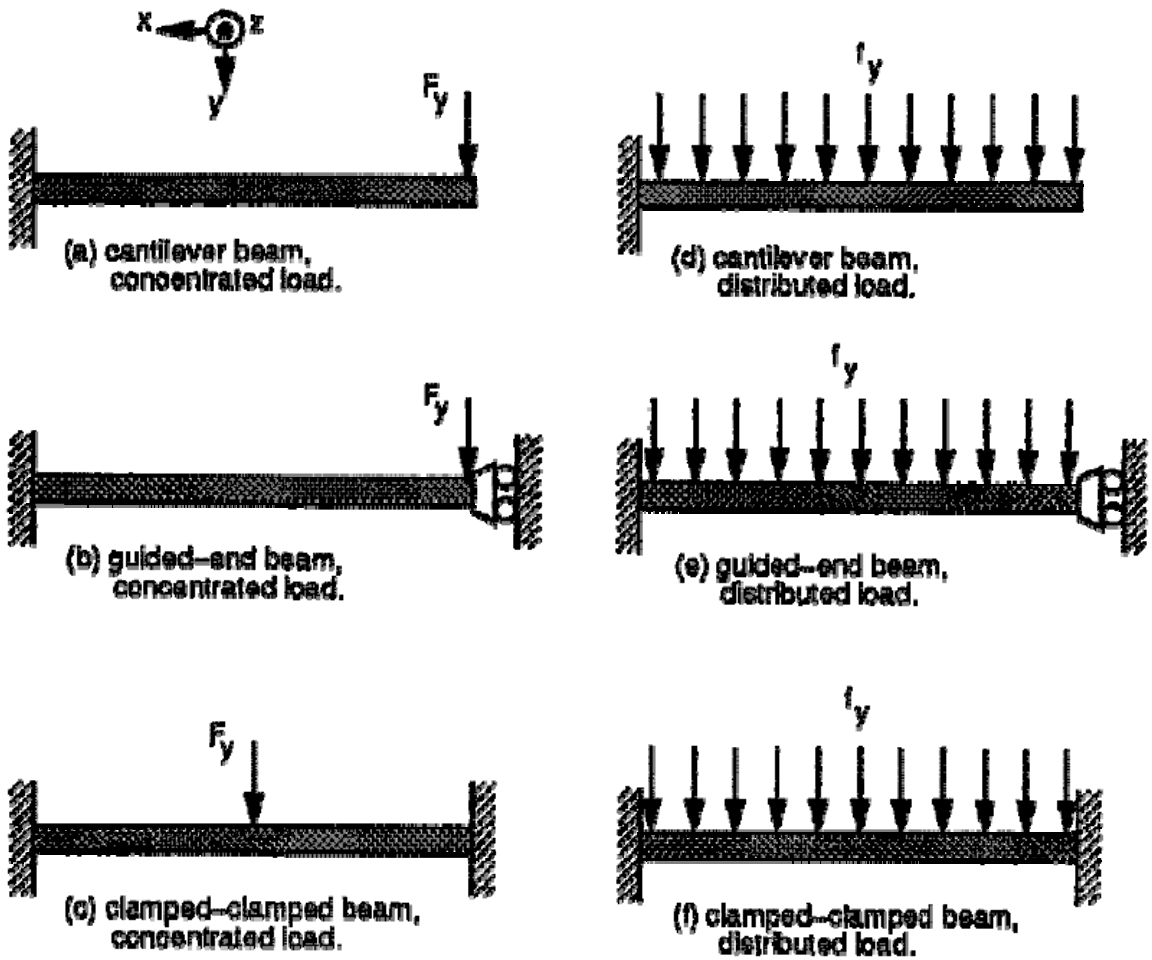
TABLE 4.1
Types of commonly used support conditions for beams and frames

Type of support	Displacement boundary conditions	Force boundary conditions
 FREE	None	All, as specified
 PINNED	$u = 0$ $w = 0$	Moment is specified
 ROLLER (vertical)	$u = 0$	Transverse force and moment are specified
 ROLLER (horizontal)	$w = 0$	Horizontal force and bending moment are specified
 FIXED or CLAMPED	$u = 0$ $w = 0$ $dw/dx = 0$	None specified

[From Reddy, Finite Element Method]

Common Loading & Boundary Conditions

- Displacement equations derived for various beams with concentrated load F or distributed load f
- Gary Fedder Ph.D. Thesis, EECS, UC Berkeley, 1994



cantilever	guided-end	fixed-fixed
$x = \frac{F_x L}{Ehw}$	$x = \frac{F_x L}{Ehw}$	$x = \frac{F_x L}{4Ehw}$
$y = 4\frac{F_y}{Eh}\frac{L^3}{w^3}$	$y = \frac{F_y}{Eh}\frac{L^3}{w^3}$	$y = \frac{1}{16}\frac{F_y}{Eh}\frac{L^3}{w^3}$
$z = 4\frac{F_z}{Ew}\frac{L^3}{h^3}$	$z = \frac{F_z}{Ew}\frac{L^3}{h^3}$	$z = \frac{1}{16}\frac{F_z}{Ew}\frac{L^3}{h^3}$

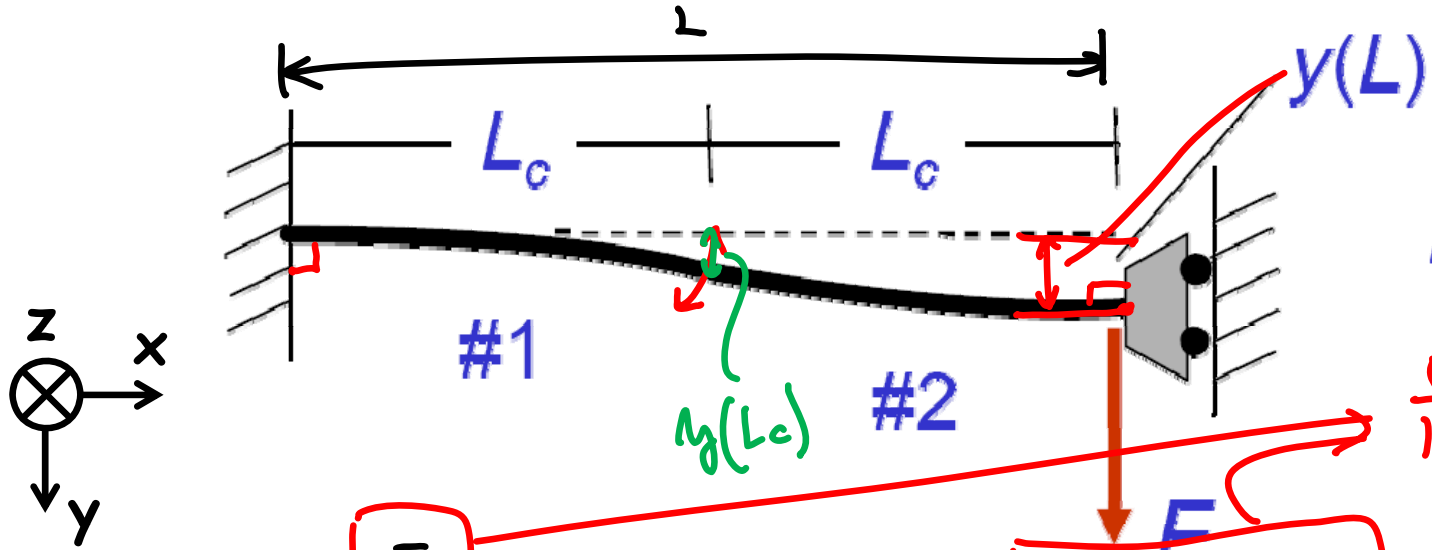
(a) Concentrated load.

cantilever	guided-end	fixed-fixed
$x = \frac{f_x L}{E}$	$x = \frac{f_x L}{E}$	$x = \frac{f_x L}{4E}$
$y = \frac{3}{2}\frac{f_y}{Eh}\frac{L^4}{w^3}$	$y = \frac{1}{2}\frac{f_y}{Eh}\frac{L^4}{w^3}$	$y = \frac{1}{32}\frac{f_y}{Eh}\frac{L^4}{w^3}$
$z = \frac{3}{2}\frac{f_z}{Ew}\frac{L^4}{h^3}$	$z = \frac{1}{2}\frac{f_z}{Ew}\frac{L^4}{h^3}$	$z = \frac{1}{32}\frac{f_z}{Ew}\frac{L^4}{h^3}$

(b) Distributed load.

Series Combinations of Springs

- For springs in series w/ one load
  - Deflections add
  - Spring constants combine like "resistors in parallel"



$L = 2L_c$

$$\frac{1}{k} = \frac{1}{k_c} + \frac{1}{k_c}$$

$$\boxed{k = k_c \| k_c}$$

$$y(L) = \boxed{\frac{F}{k}} = 2\,y(L_c) = 2\left(\frac{F}{k_c}\right) = \boxed{F\left(\frac{1}{k_c} + \frac{1}{k_c}\right)}$$

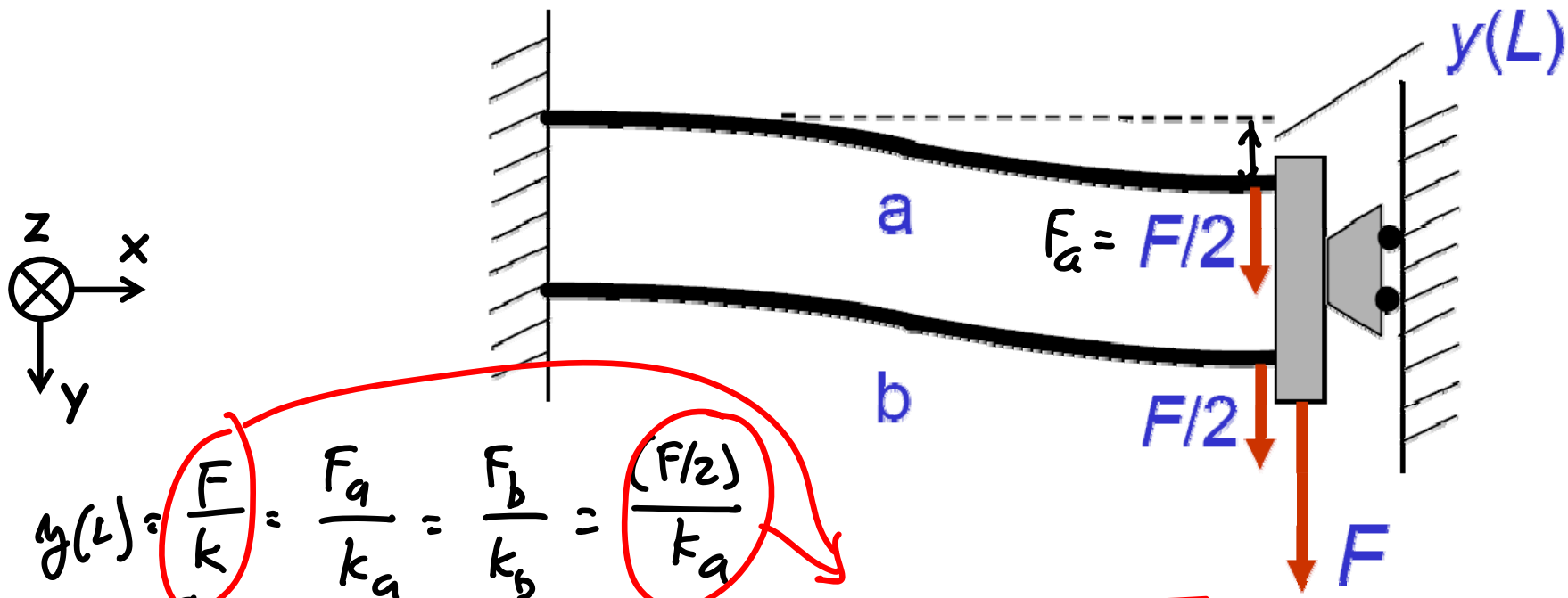
Stiffness of the Total Beam

Stiffness at the Tip of a Cantilever

Compliances add to give total compliance!

Parallel Combinations of Springs

- For springs in parallel w/ one load
  - Load is shared between the two springs
  - Spring constant is the sum of the individual spring constants



$$y(L) = \frac{F}{k} = \frac{F_a}{k_a} = \frac{F_b}{k_b} = \frac{(F/2)}{k_a}$$

Total Stiffness

Stiffness of Beam a

"

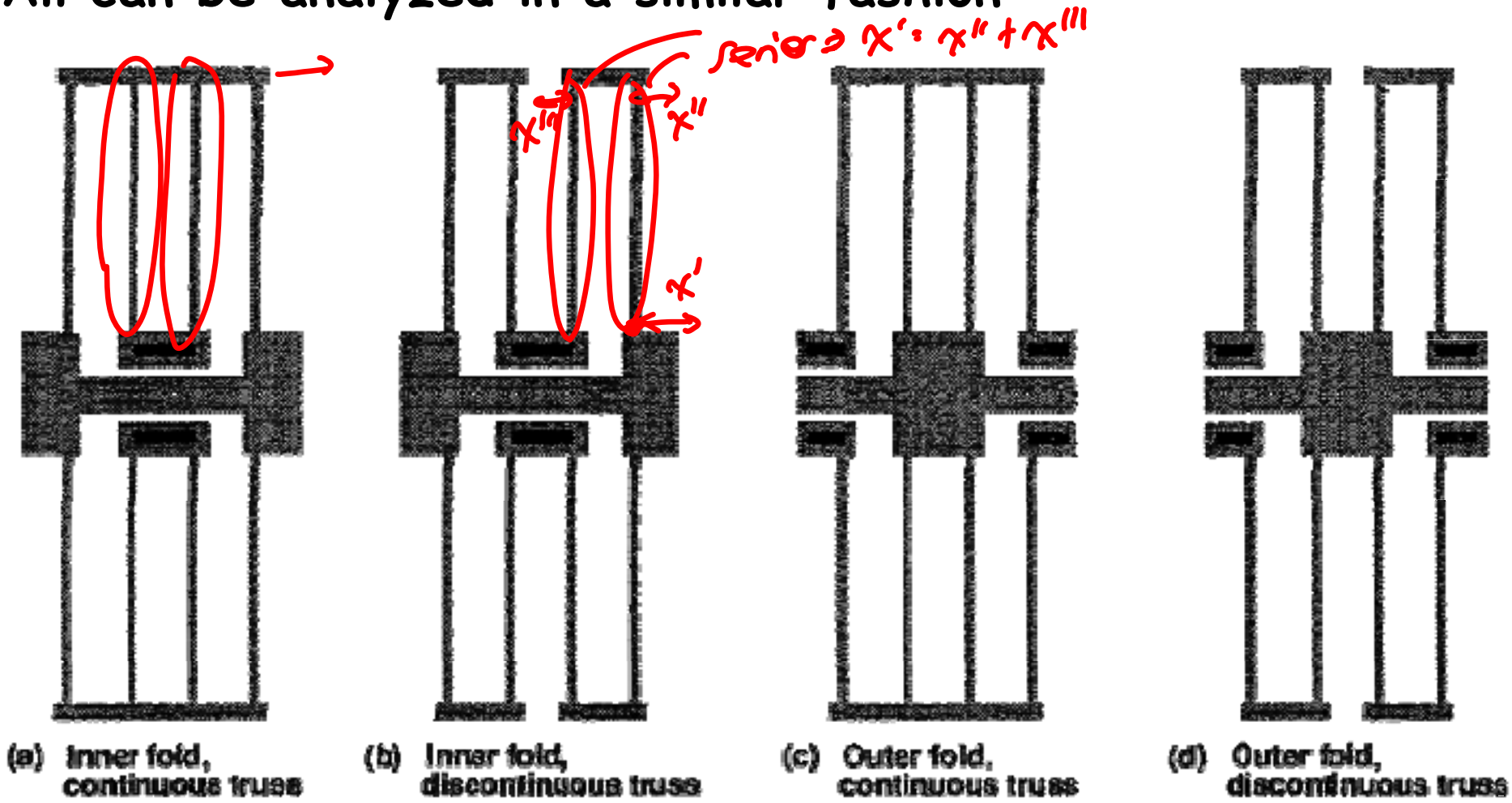
"

$$\boxed{k = 2k_a = 2k_b}$$

→ When two springs are in parallel, stiffnesses add!

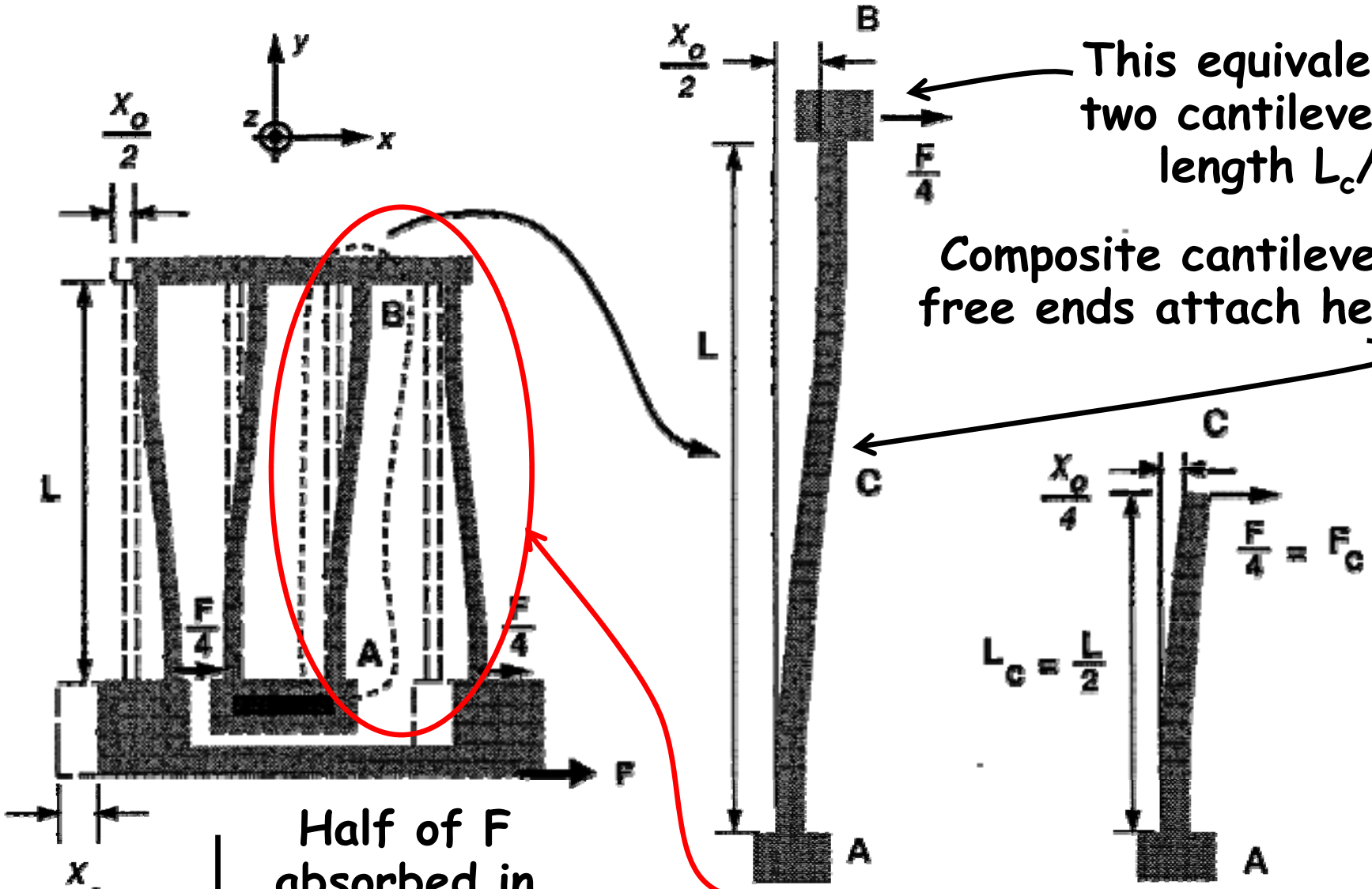
Folded-Flexure Suspension Variants

- <u>Below</u>: just a subset of the different versions
- All can be analyzed in a similar fashion



[From Michael Judy, Ph.D. Thesis, EECS, UC Berkeley, 1994]

Deflection of Folded Flexures



This equivalent to two cantilevers of length $L_c/2$

Composite cantilever free ends attach here

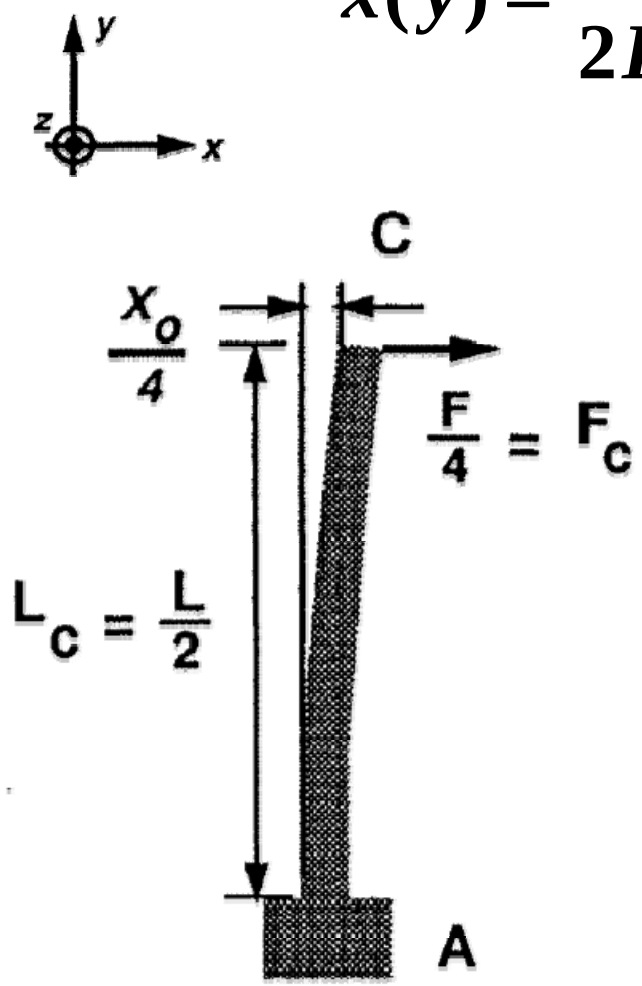
Half of F absorbed in other half (symmetrical)

4 sets of these pairs, each of which gets $\frac{1}{4}$ of the total force F

Constituent Cantilever Spring Constant

- From our previous analysis:

$$x(y) = \frac{F_c L_c}{2EI_z} y^2 \left(1 - \frac{y}{3L_c}\right) = \frac{F_c y^2}{6EI_z}(3L_c - y)$$



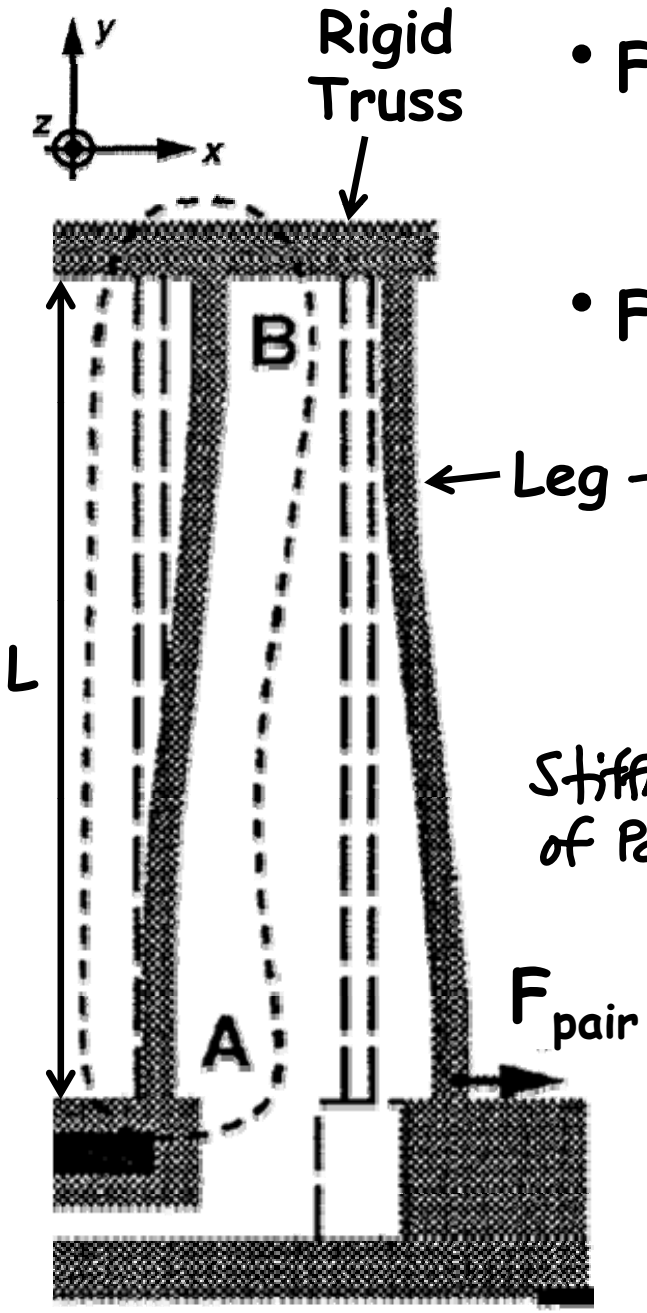
- From which the spring constant is:

$$k_c = \frac{F_c}{x(L_c)} = \frac{3EI_z}{L_c^3}$$

- Inserting $L_c = L/2$

$$\boxed{k_c = \frac{3EI_z}{(L/2)^3} = \frac{24EI_z}{L^3}}$$

Overall Spring Constant



- Four pairs of clamped-guided beams
  - In each pair, beams bend in series
  - (Assume trusses are inflexible)
- Force is shared by each pair → $F_{pair} = F/4$

total force on shuttle

← Leg → Displacement of two legs add

↳ thus, springs are in series:

$$x = \frac{F_{pair}}{k_{pair}} = \frac{F_{pair}}{(k_{leg} \| k_{leg})} = \left(\frac{F}{4}\right)\left(\frac{1}{k_{leg}} + \frac{1}{k_{leg}}\right)$$

Stiffness of Pair

From before: $k_{leg} = k_c \| k_c = \frac{k_c}{2}$

Thus:

$$x = \left(\frac{F}{4}\right)\left(\frac{2}{k_c} + \frac{2}{k_c}\right) = \frac{F}{k_c} = \frac{F}{k_{tot}}$$

$$\Rightarrow \boxed{k_{tot} = k_c = \frac{24 E I_z}{L^3}}$$