- **Announcements**:
- Lecture schedule for the next 4 weeks:

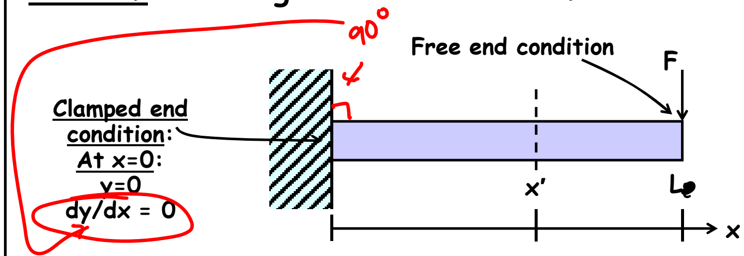
Monday	Tuesday	Wednesday	Thursday	Friday
Oct. 5 Lecture 5-6:30 241 Cory	6 Lecture 3:30-5 Moffitt	7	8 Discuss 3:30-5 Moffitt	9
12 Lecture 5-6:30 241 Cory	13	14	15 Discuss 3:30-5 Moffitt	16
19 Lecture 5-6:30 241 Cory	20 Lecture 3:30-5 Moffitt	21	22 Lecture 3:30-5 Moffitt	23
26 Discuss 3:30-5 Moffitt	27	28	29 Lecture 3:30-5 Moffitt	30
Nov. 2 Lecture 5-6:30 241 Cory	3 Midterm 3:30-5 Moffitt	4	5 Lecture 3:30-5 Moffitt	6

- Midterm Exam:
  - Tu, Nov. 3, during regular lecture
  - 1.5 hours, but might go longer

- **Today**:
- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients

- ----------------------------------

Problem: Bending a Cantilever Beam



- **Objective**: Find relation between tip deflection $y(x=L_c)$ and applied load F
- **Assumptions**:
  1. Tip deflection is small compared with beam length
  2. Plane sections (normal to beam's axis) remain plane and normal during bending, i.e., "pure bending"
  3. Shear stresses are negligible

Maximum deflection @ $x=L$:

$$w_{max} = \left(\frac{L^3}{3EI}\right)F \rightarrow F = \left(\frac{3EI}{L^3}\right)w(x=L)$$

$$= k_c w(x=L)$$

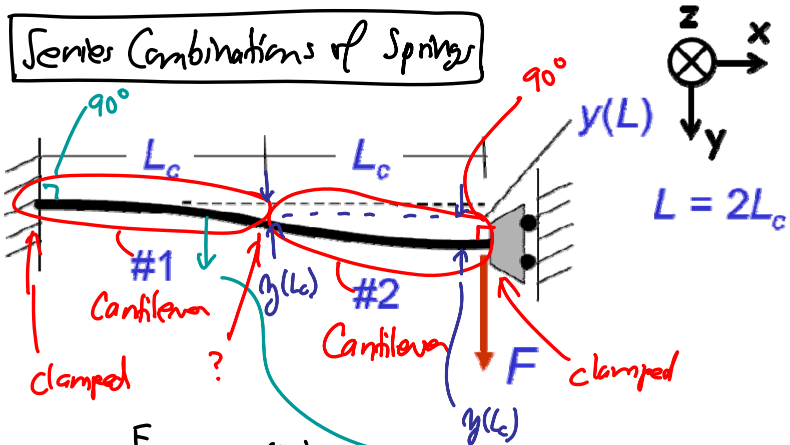
"cantilever"

where $k_c = \frac{3EI}{L^3} \triangleq$ stiffness @ location $x=L$ "cantilever"

in general, stiffness is a function of location

$\left[I = \frac{1}{12}Wh^3\right] \Rightarrow \boxed{k_c = \frac{1}{4}EW\frac{h^3}{L^3}}$

Series Combinations of Springs



$y(L) = \frac{F}{k} = 2y(L_c)$

stiffness of the whole thing

$= 2\left(\frac{F}{k_c}\right) = F\underbrace{\left(\frac{1}{k_c}+\frac{1}{k_c}\right)}$ compliances add!

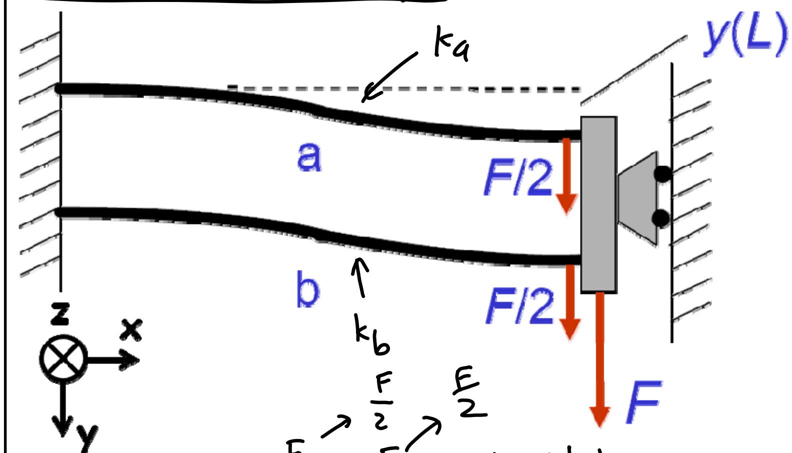
stiffness of a cantilever of length L_c

$\boxed{\frac{1}{k} = \frac{1}{k_c}+\frac{1}{k_c}} \rightarrow \boxed{k = k_c \| k_c}$

Like "capacitors in series"

Definition for "$\|$": $A\|B = \frac{1}{\frac{1}{A}+\frac{1}{B}} = \frac{AB}{A+B}$

Parallel Combination of Springs



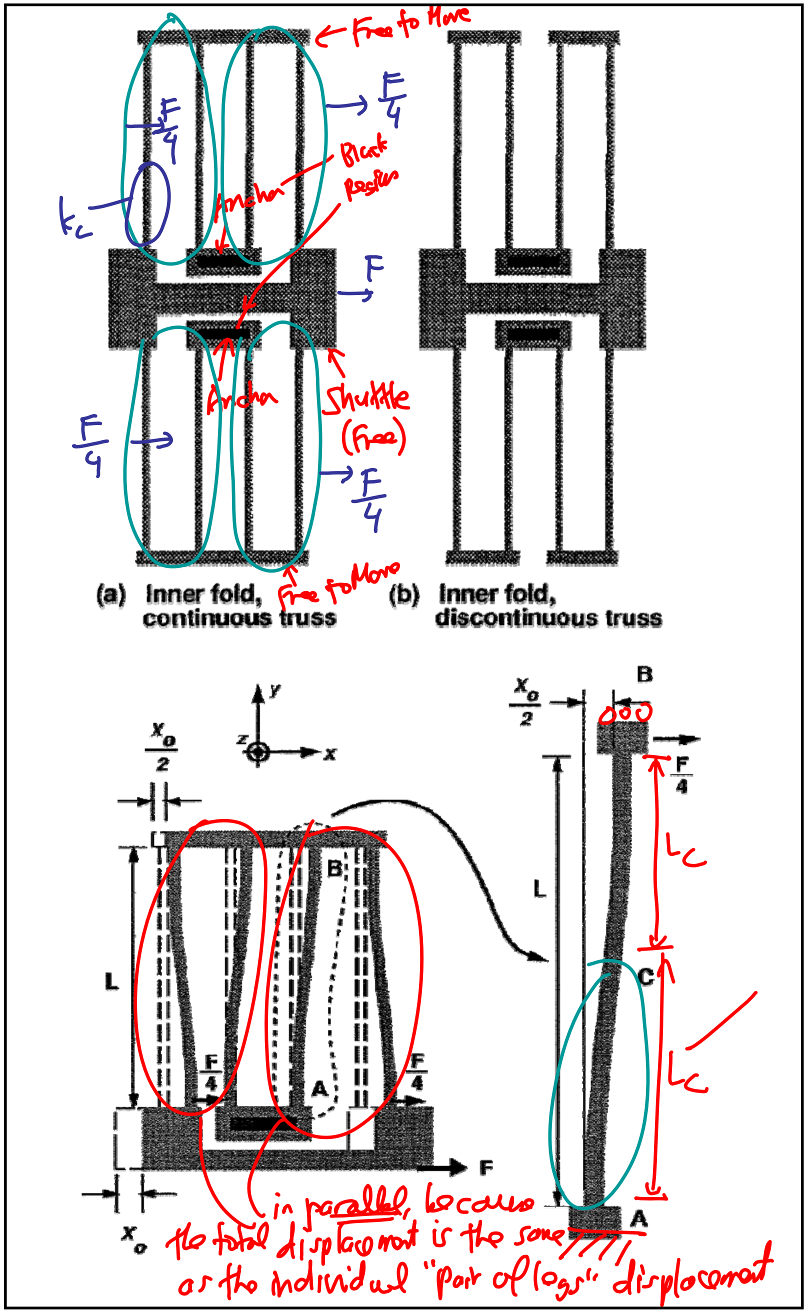
$y(L) = \frac{F}{k} = \frac{F_a}{k_a} = \frac{F_b}{k_b} = \left(\frac{F}{2}\right)\left(\frac{1}{k_a}\right)$, with $F_a = \frac{F}{2}$, $F_b = \frac{F}{2}$

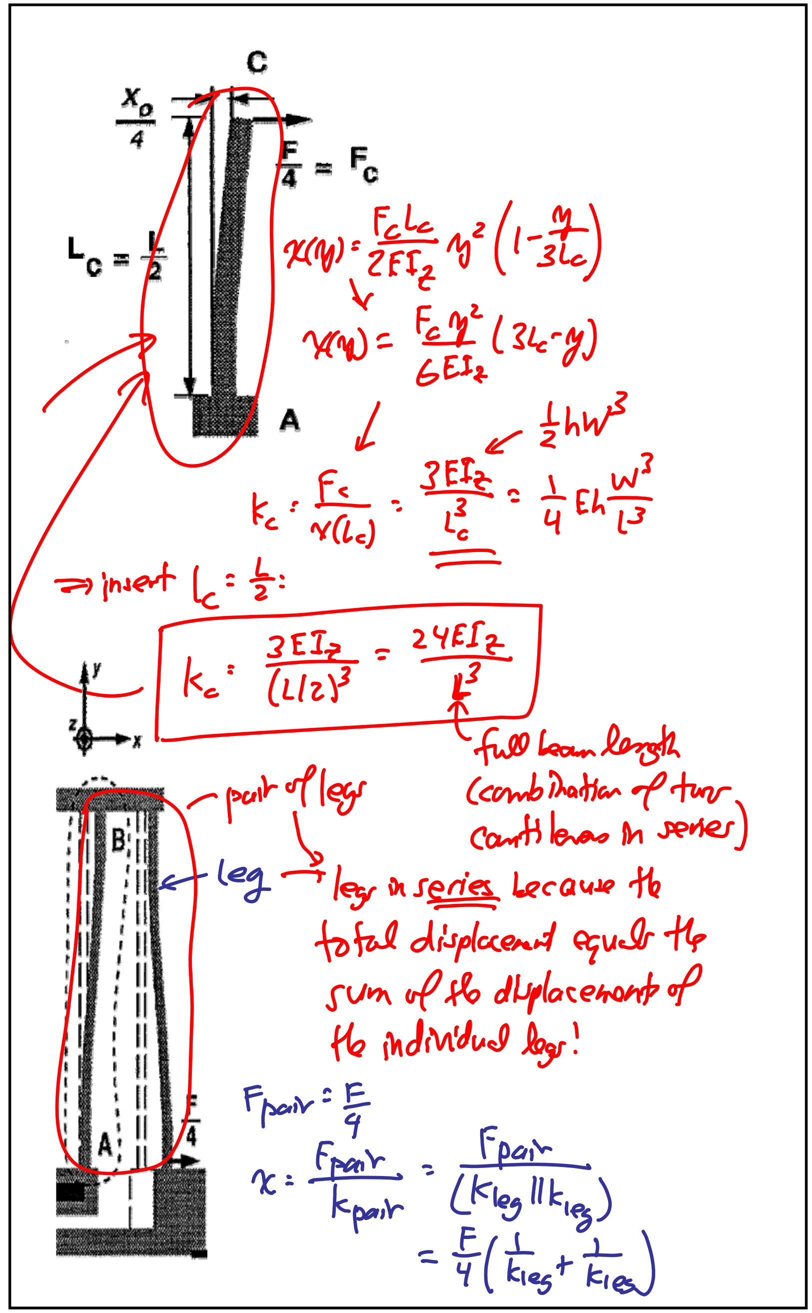
of the whole thing

we want this!

$\boxed{k = 2k_a}$

Again, like capacitors in parallel!





$$x(y) = \frac{F_c L_c}{2EI_z} y^2 \left(1 - \frac{y}{3L_c}\right)$$

$$x(y) = \frac{F_c y^2}{6EI_z}(3L_c - y)$$

$$k_c = \frac{F_c}{x(L_c)} = \underline{\frac{3EI_z}{L_c^3}} = \frac{1}{4}Eh\frac{W^3}{L^3}$$

($I_z = \frac{1}{12}hW^3$)

$\Rightarrow$ insert $L_c = \frac{L}{2}$:

$$\boxed{k_c = \frac{3EI_z}{(L/2)^3} = \frac{24EI_z}{L^3}}$$

full beam length (combination of two cantilevers in series)

legs in series because the total displacement equals the sum of the displacements of the individual legs!

$F_{pair} = \frac{F}{4}$

$$x = \frac{F_{pair}}{k_{pair}} = \frac{F_{pair}}{(k_{leg} \| k_{leg})} = \frac{F}{4}\left(\frac{1}{k_{leg}} + \frac{1}{k_{leg}}\right)$$

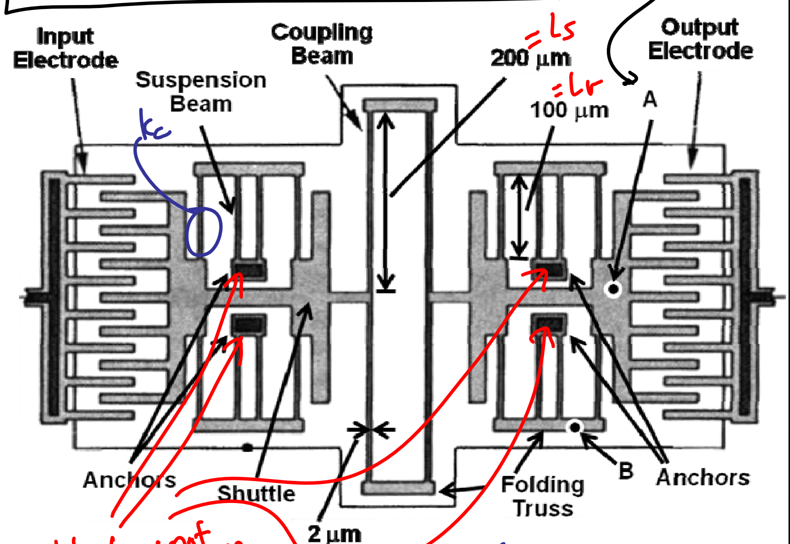
From before: $k_{leg} = k_c \| k_c = \frac{k_c}{2}$

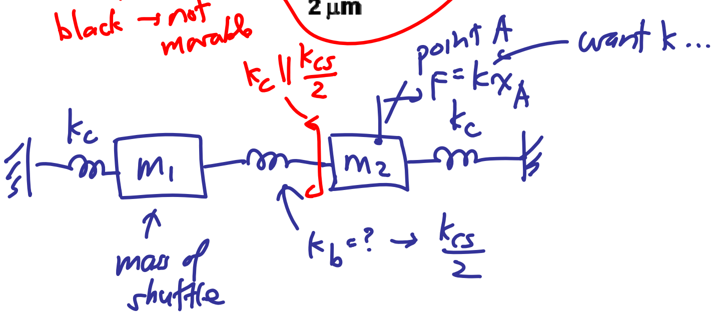
Thus:

$$x = \left(\frac{F}{4}\right)\left(\frac{2}{k_c} + \frac{2}{k_c}\right) = \frac{F}{k_c} = \frac{F}{k_{tot}}$$

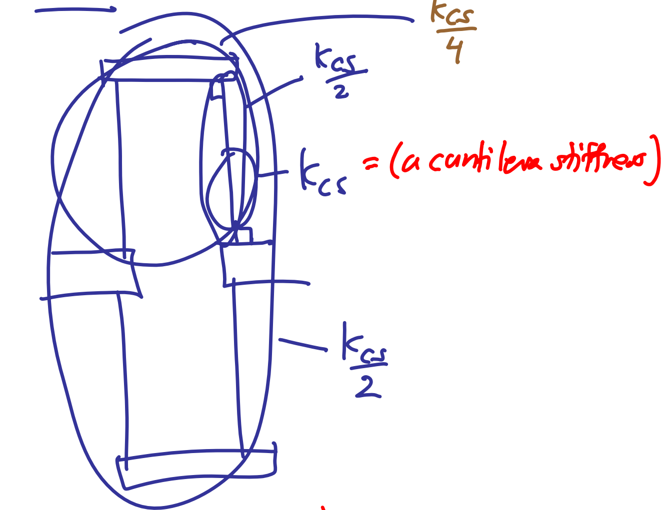
$$\boxed{k_{tot} = k_c = \frac{24EI_z}{L^3}}$$

How about this? → Find the stiffness @ point A





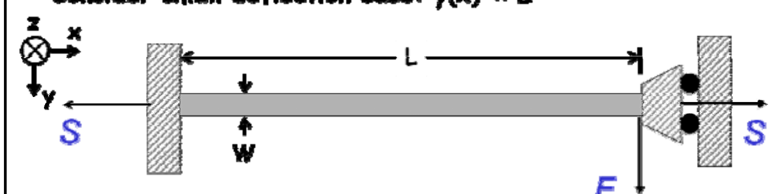
Get k_b:



$$\therefore k_{xA} = k_c + k_c \| \frac{k_{cs}}{2}$$

where: $k_c = \frac{24EI_z}{L_r^3}$ $\quad k_{cs} = \frac{24EI_z}{L_s^3}$

- Important case for MEMS suspensions, since the thin films comprising them are often under residual stress
- Consider small deflection case: $y(x) \ll L$



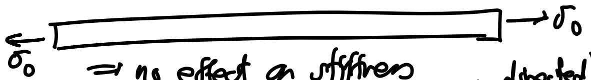
Governing differential equation: (Euler Beam Equation)

$$EI_z \frac{d^4 y}{dx^4} - S\frac{d^2 y}{dx^2} = \underbrace{F\delta(x-L)}_{\text{Unit impulse @ } x=L}$$

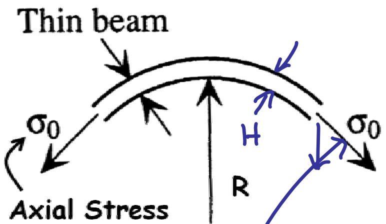
Axial Load

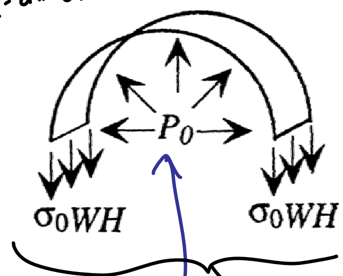
Heuristic Derivation for the Euler Beam Equation

Consider first a straight beam (unflexed) under axial stress:



$\Rightarrow$ no effect on stiffness $\hat{z}$-directed (or w-directed)





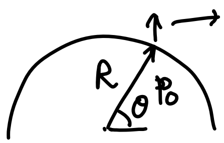
w/ beam bent, σ_0 has a z-directed component → affects stiffness!

Upward pressure P_0 to counteract the downward force from), to keep everything in static equilibrium

For ease of analysis, assume the beam is bent to angle π

Downward Vertical Force $= 2\sigma_0 WH$

Upward Force due to P_0:



$\rightarrow P_y(\theta) = P_0 \sin\theta$

$$F_u = \int_0^{\pi} (P_0 \sin\theta) W (R\,d\theta)$$
$$= -P_0 W R \cos\theta \Big|_0^{\pi}$$
$$= 2RWP_0$$