

Solution: Use Principle of Virtual Work

- In an energy-conserving system (i.e., elastic materials), the energy stored in a body due to the quasi-static (i.e., slow) action of surface and body forces is equal to the work done by these forces ...
- Implication:** if we can formulate **stored energy** as a function of the deformation of a mechanical object, then we can determine how an object responds to a force by determining the shape the object must take in order to **minimize the difference U** between the stored energy and the work done by the forces:

$$U = \text{Stored Energy} - \text{Work Done}$$

- Key idea:** we don't have to reach $U = 0$ to produce a very useful, approximate *analytical* result for load-deflection

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Shear Strain Energy

$$W_{\text{shear}} = \frac{3(EI_z)^2}{4GWh} \int_0^L \left(\frac{d^3 y}{dx^3} \right)^2 dx$$

Shear Modulus

- See W.C. Albert, "Vibrating Quartz Crystal Beam Accelerometer," Proc. ISA Int. Instrumentation Symp., May 1982, pp. 33-44

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Applying the Principle of Virtual Work

- Basic Procedure:**
 - Guess the form of the beam deflection under the applied loads
 - Vary the parameters in the beam deflection function in order to minimize:

Sum strain energies

Assumes point load

$$U = \sum_j W_j - \sum_i F_i u_i$$

Displacement at point load

Find minima by simply setting derivatives to zero

- See Senturia, pg. 244, for a general expression with distributed surface loads and body forces

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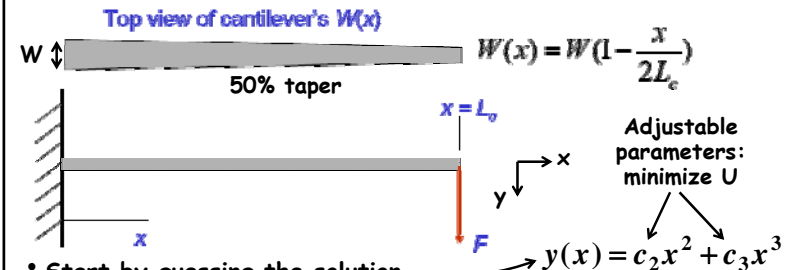
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Example: Tapered Cantilever Beam

- Objective:** Find an expression for displacement as a function of location x under a point load F applied at the tip of the free end of a cantilever with tapered width $W(x)$



- Start by guessing the solution
 - It should satisfy the boundary conditions
 - The strain energy integrals shouldn't be too tedious
 - This might not matter much these days, though, since one could just use matlab or mathematica

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Strain Energy And Work By F

$U = \mathcal{W}_{\text{bend}} - F \cdot y(L_e)$

$\mathcal{W}_{\text{bend}} = \frac{1}{2} E \int_0^{L_e} I_z(x) \left(\frac{d^2 y}{dx^2} \right)^2 dx$ (Bending Energy)

$I_z(x) = \frac{W(x)h^3}{12}$

$W(x) = W \left(1 - \frac{x}{2L_e} \right)$

$\frac{d^2 y}{dx^2} = 2c_2 + 6c_3 x$ (Using our guess)

Tip Deflection

$= \frac{1}{24} E W h^3 \int_0^{L_e} \left(1 - \frac{x}{2L_e} \right) (2c_2 + 6c_3 x)^2 dx - F(c_2 L_e^2 + c_3 L_e^3)$

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Find c_2 and c_3 That Minimize U

- Minimize U → basically, find the c_2 and c_3 that brings U closest to zero (which is what it would be if we had guessed correctly)
- The c_2 and c_3 that minimize U are the ones for which the partial derivatives of U with respect to them are zero:

$$\frac{\partial U}{\partial c_2} = 0 \quad \frac{\partial U}{\partial c_3} = 0$$

- Proceed:
 - First, evaluate the integral to get an expression for U:

$$U = E W h^3 \left\{ \frac{5c_3^2}{16} L_e^3 + \frac{c_2 c_3}{3} L_e^2 + \frac{c_2^2}{8} L_e \right\} - F(c_2 L_e^2 + c_3 L_e^3)$$

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Minimize U (cont)

- Evaluate the derivatives and set to zero:

$$\frac{\partial U}{\partial c_2} = 0 = \left(\frac{E W h^3}{3} c_3 - F \right) L_e^2 + \left(\frac{E W h^3}{4} c_2 \right) L_e$$

$$\frac{\partial U}{\partial c_3} = 0 = \left(\frac{5}{8} E W h^3 c_3 - F \right) L_e^3 + \left(\frac{E W h^3}{3} c_2 \right) L_e^2$$

- Solve the simultaneous equations to get c_2 and c_3 :

$$c_2 = \left(\frac{84}{13} \right) \frac{F L_e}{E W h^3} \quad c_3 = - \left(\frac{24}{13} \right) \frac{F}{E W h^3}$$

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The Virtual Work-Derived Solution

- And the solution:

$$y(x) = \left(\frac{24F}{13E W h^3} \right) \left(\left(\frac{7}{2} \right) L_e - x \right) x^2$$

- Solve for tip deflection and obtain the spring constant:

$$y(L_e) = \left(\frac{24F}{13E W h^3} \right) \left(\frac{5}{2} \right) L_e^3 \quad k_e = F / y(L_e) = \left(\frac{13E W h^3}{60 L_e^3} \right)$$

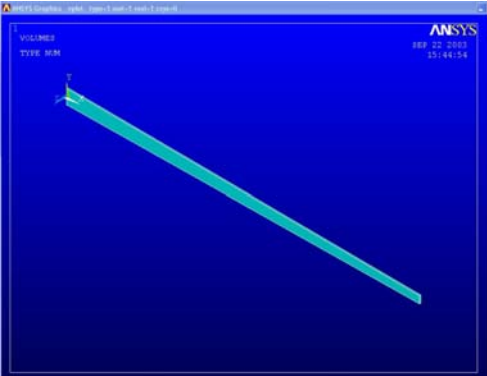
- Compare with previous solution for constant-width cantilever beam (using Euler theory):

$$y(L_e) = \left(\frac{4F}{E W h^3} \right) L_e^3 \longrightarrow 13\% \text{ smaller than tapered-width case}$$

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Comparison With Finite Element Simulation

- Below: ANSYS finite element model with
 $L = 500 \mu\text{m}$ $W_{\text{base}} = 20 \mu\text{m}$ $E = 170 \text{ GPa}$
 $h = 2 \mu\text{m}$ $W_{\text{tip}} = 10 \mu\text{m}$



- Result:** (from static analysis)
 $k = 0.471 \mu\text{N/m}$
- This matches the result from energy minimization to 3 significant figures

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Need a Better Approximation?

- Add more terms to the polynomial
- Add other strain energy terms:
 - Shear: more significant as the beam gets shorter
 - Axial: more significant as deflections become larger
- Both of the above remedies make the math more complex, so encourage the use of math software, such as Mathematica, Matlab, or Maple
- Finite element analysis is really just energy minimization
- If this is the case, then why ever use energy minimization analytically (i.e., by hand)?
 - Analytical expressions, even approximate ones, give insight into parameter dependencies that FEA cannot
 - Can compare the importance of different terms
 - Should use in tandem with FEA for design

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