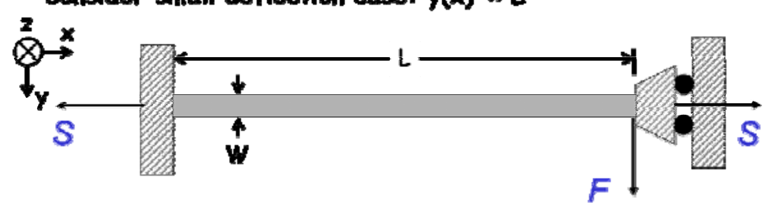
- **Announcements**:
- Lecture schedule for the next 4 weeks:

Monday	Tuesday	Wednesday	Thursday	Friday
Oct. 5 Lecture 5-6:30 241 Cory	6 Lecture 3:30-5 Moffitt	7	8 Discuss 3:30-5 Moffitt	9
12 Lecture 5-6:30 241 Cory	13	14	15 Discuss 3:30-5 Moffitt	16
19 Lecture 5-6:30 241 Cory	20 Lecture 3:30-5 Moffitt	21	22 Lecture 3:30-5 Moffitt	23
26 Discuss 3:30-5 Moffitt	27	28	29 Lecture 3:30-5 Moffitt	30
Nov. 2 Lecture 5-6:30 241 Cory	3 Midterm 3:30-5 Moffitt	4	5 Lecture 3:30-5 Moffitt	6

- Midterm Exam:
  - Tu, Nov. 3, during regular lecture
  - 1.5 hours, but might go longer
- HW#5: Get it off the web later today

- **Today**:
- Reading: Senturia, Chpt. 9
- Lecture Topics:
  - Bending of beams
  - Cantilever beam under small deflections
  - Combining cantilevers in series and parallel
  - Folded suspensions
  - Design implications of residual stress and stress gradients
  - Energy Methods
    - Virtual Work
    - Energy Formulations
    - Tapered Beam Example
    - Estimating Resonance Frequency
- ----------------------------------
- **Last Time**:

- Important case for MEMS suspensions, since the thin films comprising them are often under residual stress
- Consider small deflection case: $y(x) \ll L$



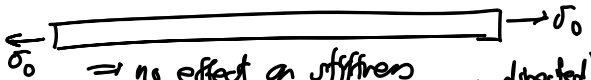
Governing differential equation: (Euler Beam Equation)

$$EI_z \frac{d^4y}{dx^4} - S\frac{d^2y}{dx^2} = \underbrace{F\delta(x-L)}_{\text{Unit impulse @ x=L}}$$

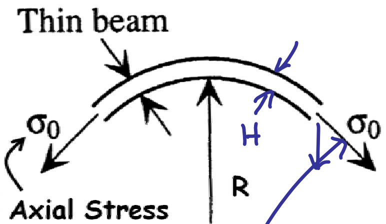
Axial Load

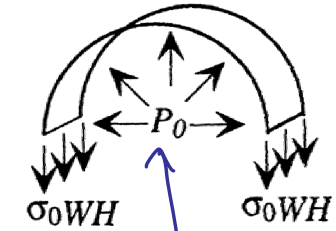
Heuristic Derivation for the Euler Beam Equation

Consider first a straight beam (unflexed) under axial stress:



$\Rightarrow$ no effect on stiffness
$\hat{z}$-directed (or w-directed)





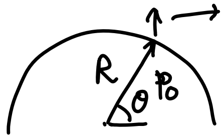
w/ beam bent, σ_0 has a z-directed component $\rightarrow$ affects stiffness!

Upward pressure P_0 to counteract the downward force from σ_0 to keep everything in static equilibrium

For ease of analysis, assume the beam is bent to angle π

Downward Vertical Force $= 2\sigma_0 WH$

Upward Force due to P_0:



$\uparrow \rightarrow P_y(\theta) = P_0 \sin\theta$

$$F_u = \int_0^{\pi} (P_0 \sin\theta) W (R d\theta)$$
$$= -P_0 W R \cos\theta \Big|_0^{\pi}$$
$$= 2RWP_0$$

[Equilibrium] $\rightarrow 2RWP_0 = 2\sigma_0 WH \rightarrow P_0 = \frac{\sigma_0 H}{R}$

$\left[q_0 = \frac{\text{beam load}}{\text{unit length}} = P_0 W, \quad \frac{1}{R} = \frac{d^2 w}{dx^2} \right] \Rightarrow$

beam displacement

$$q_0 = \sigma_0 WH \frac{d^2 w}{dx^2}$$

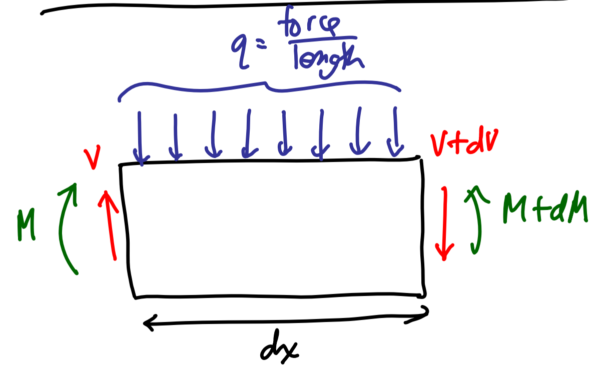
$\rightarrow$ generalizes to the case of small displacements & small angles

Using the differential beam bending equation:

$$\frac{d^2 w}{dx^2} = -\frac{M}{EI} \rightarrow \frac{d^4 w}{dx^4} = \frac{q}{EI}$$

$q \leftarrow \frac{\text{load}}{\text{unit length}}$

Relationships Between Forces on a Fully Loaded Differential Beam Element



[Total static equilibrium] $\rightarrow$ total force $= 0$

$$F_T = \text{total force} = q\,dx + (\cancel{V} + dV) - \cancel{V} = 0$$

$$\therefore \boxed{\frac{dV}{dx} = -q} \qquad (1)$$

→ also, total moment wrt to the left hand edge = 0

$$M_T = (\cancel{M}+dM) - \cancel{M} - (V+\cancel{dV})dx - \frac{q\,\cancel{dx}}{2}\cancel{dx} = 0$$

$$\int_0^{dx}(q\,du)u = \frac{1}{2}q\,dx^2 \checkmark$$

[neglect product of differentials] →

$$dM - V\,dx = 0 \rightarrow \boxed{\frac{dM}{dx} = V} \qquad (2)$$

Using (1) & (2):

$$+\quad \frac{d^2M}{dx^2} = \frac{dV}{dx} = -q$$

* →

$$EI\frac{d^4w}{dx^4} = q + q_0$$

external load

equiv. load accounts for axial stress contributions to the bending stiffness

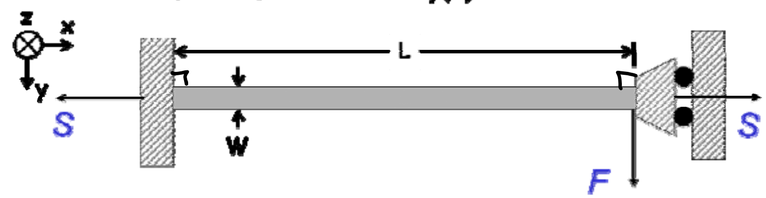
$$\left\{q_0 = \sigma_0 WH\frac{d^2w}{dx^2}\right\} \Rightarrow \boxed{EI\frac{d^4w}{dx^4} - \underbrace{(\sigma_0 WH)}_{\text{tension in the beam} = S}\frac{d^2w}{dx^2} = q}$$

S: a force

Euler Beam Equation

Clamped-Guided Beam Under Axial Load

- Important case for MEMS suspensions, since the thin films comprising them are often under residual stress
- Consider small deflection case: $y(x) \ll L$



Governing differential equation: (Euler Beam Equation)

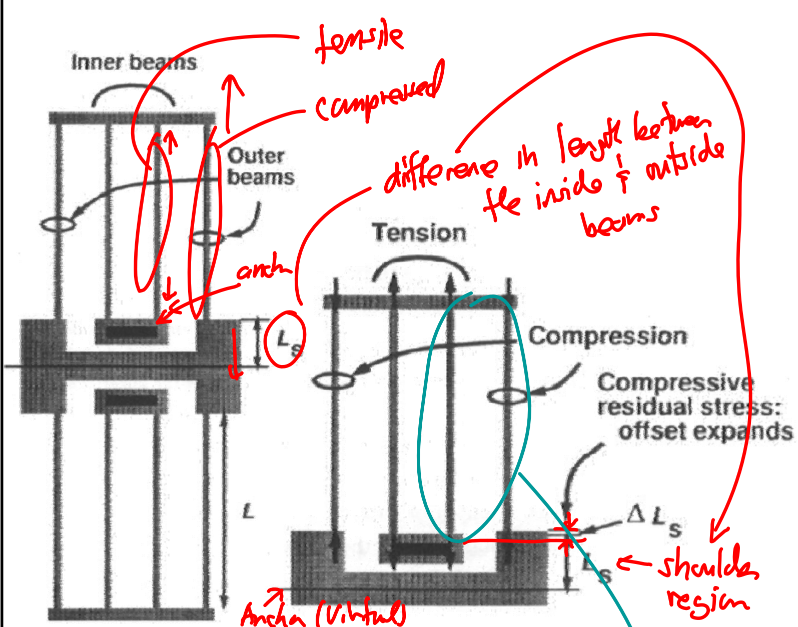
$$EI_z\frac{d^4y}{dx^4} - \underbrace{S}_{\text{Axial Load}}\frac{d^2y}{dx^2} = \underbrace{F\delta(x-L)}_{\text{Unit impulse @ x=L}}$$

- Can solve the ODE using standard methods
  - Senturia, pp. 232-235: solves ODE for case of point load on a clamped-clamped beam (which defines B.C.'s)
  - For solution to the clamped-guided case: see S. Timoshenko, Strength of Materials II: Advanced Theory and Problems, McGraw-Hill, New York, 3rd Ed., 1955
- Result from Timoshenko:

$S > 0$ (tension)
$$k^{-1} = \frac{pL - 2\tanh(pL/2)}{p|S|} = \frac{y(x=L)}{F}$$

$S < 0$ (compression)
$$k^{-1} = \frac{-pL + 2\tan(pL/2)}{p|S|} = \frac{y(x=L)}{F}$$

where $p = \sqrt{\frac{|S|}{EI_z}}$



① If the poly-si strain is ϵ_r, then shoulder expands by $\Delta L_s = \epsilon_r L_s$

② This then applies a load to the beams, $\Delta L = \Delta L_s$.

③ Beam Strain:

$$\epsilon_b = \frac{\Delta L}{L} = \frac{\Delta L_s}{L} = \pm \epsilon_r \frac{L_s}{L}$$

Stress Force: $S = \pm E \epsilon_r \left(\frac{L_s}{L}\right) Wh$

Spring Constants become:

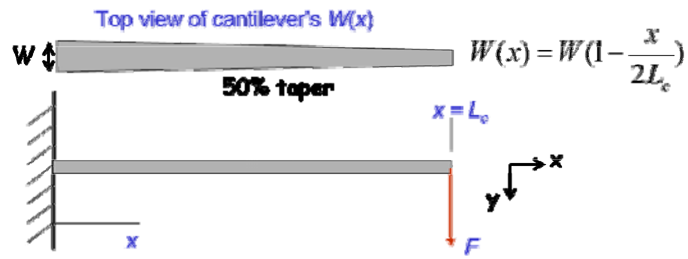
$$k = 4(k_{com}^{-1} + k_{ten}^{-1})^{-1}$$

4 pairs in parallel

$$k = 4\left[\frac{-pL + 2\tan(pL/2)}{p|S|} + \frac{pL - 2\tanh(pL/2)}{p|S|}\right]^{-1}$$

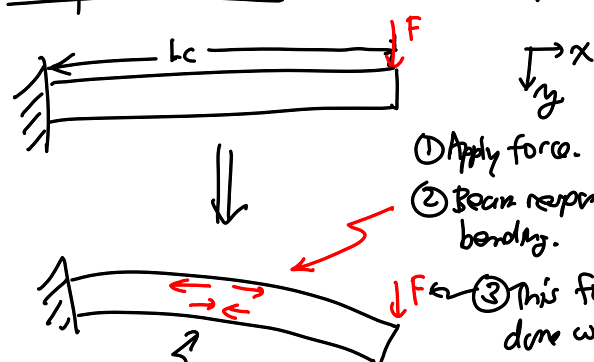
More General Geometries

- Euler-Bernoulli beam theory works well for simple geometries
- But how can we handle more complicated ones?
- **Example**: tapered cantilever beam
- **Objective**: Find an expression for displacement as a function of location x under a point load F applied at the tip of the free end of a cantilever with tapered width W(x)



Looked @ "Principle of Virtual Work" slide (cf Module 9)
→ More Visual Description

Same problem as before: Take a beam & apply a force:



① Apply force.

② Beam responds by bending.

③ This force has done work:

$W = F \cdot y(L_c)$

④ Strain generated
→ So the beam has received an influx of stored energy. magnitude determined by the deformed shape!

(5) Then:

$U = \text{Stored Energy} - \text{Work Done} \rightarrow 0$

When we choose the right shape!

This is how we get the beam's response to F!

Fundamentals: Energy Density

General Definition of Work:

$$W(q_1) = \int_0^{q_1} e(q)\,dq$$

q = displacement
e = effort

In EE: $W(Q) = \int_0^Q \frac{Q}{C}\,dQ$

Strain Energy Density

value of strain @ position (x,y,z)

$$w = \int_0^{\epsilon_x} \sigma_x\, d\epsilon_x \qquad [J/m^3] = Pa$$

$\sigma_x(\epsilon_x)$ → relates stress to strain @ position (x,y,z)

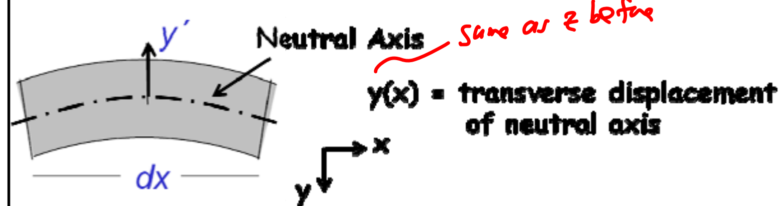
$$[\sigma_x = E\epsilon_x] \Rightarrow w = \int_0^{\epsilon_x} E\epsilon_x\, d\epsilon_x = \frac{1}{2}E\epsilon_x^2$$

Total Strain Energy [J]: (In 3D)

volume

$$W = \iiint \left(\frac{1}{2}E\left(\epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2\right) + \frac{1}{2}G\left(\gamma_{xy}^2 + \gamma_{xz}^2 + \gamma_{yz}^2\right)\right) dV$$

Bending Energy Density



same as before

First, find the bending energy dW_{bend} in an infinitesimal length dx:

$$dW_{bend} = W\,dx \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{2}E\epsilon_x^2(y')\,dy'$$

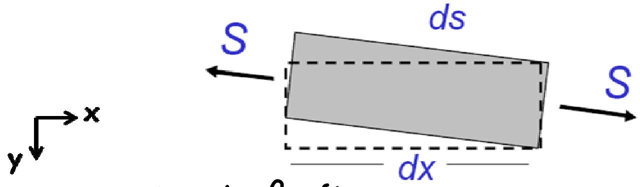
$$\left[\frac{1}{R} = \frac{d^2y}{dx^2},\ \epsilon_x = \frac{y'}{R}\right] \Rightarrow \epsilon_x(y') = y'\frac{d^2y}{dx^2}$$

$$dW_{bend} = W\,dx \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{2}E\left[y'\frac{d^2y}{dx^2}\right]^2 dy'$$

$$= \frac{1}{2}E\underbrace{\left(\frac{Wh^3}{12}\right)}_{I_z}\left(\frac{d^2y}{dx^2}\right)^2 dx$$

$$\therefore \boxed{W_{bend} = \frac{1}{2}EI_z \int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx}$$

Energy Due to Axial Load



⇒ energy related to lengthening

$$ds = \left[(dx)^2 + (dy)^2\right]^{1/2} = dx\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2}$$

Binomial Theorem $\approx dx\left[1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2\right]$

$$\therefore \epsilon_x = \frac{ds - dx}{dx} = \frac{1}{2}\left(\frac{dy}{dx}\right)^2$$

$$\left[dW_{axial} = S\epsilon_x dx = \frac{1}{2} S \left(\frac{dy}{dx}\right)^2 dx\right]$$

$$\boxed{W_{axial} = \frac{1}{2} S \int_0^L \left(\frac{dy}{dx}\right)^2 dx}$$

Axial Strain Energy

⇒ look @ shear strain energy in your module.

- Go through Module 9 pages 10-18.