EE 245: Introduction to MEMS
Lecture 19: Resonance Frequency II

- **Announcements**:
- Lecture schedule for the next 4 weeks:

Monday	Tuesday	Wednesday	Thursday	Friday
Oct. 5 Lecture 5-6:30 241 Cory	6 Lecture 3:30-5 Moffitt	7	8 Discuss 3:30-5 Moffitt	9
12 Lecture 5-6:30 241 Cory	13	14	15 Discuss 3:30-5 Moffitt	16
19 Lecture 5-6:30 241 Cory	20 Lecture 3:30-5 Moffitt	21	22 Lecture 3:30-5 Moffitt	23
26 Discuss 3:30-5 Moffitt	27	28	29 Lecture 3:30-5 Moffitt	30
Nov. 2 Lecture 5-6:30 241 Cory	3 Midterm 3:30-5 Moffitt	4	5 Lecture 3:30-5 Moffitt	6

- Midterm Exam:
  - Tu, Nov. 3, during regular lecture
  - 1.5 hours, but might go longer

- Reading: Senturia, Chpt. 10: §10.5, Chpt. 19
- Lecture Topics:
  - Estimating Resonance Frequency
  - Lumped Mass-Spring Approximation
  - ADXL-50 Resonance Frequency
  - Distributed Mass & Stiffness
  - Folded-Beam Resonator
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- **Last Time**:

$$K_{max} = \int_0^L \frac{1}{2}\rho W h\, dx\, v^2(x,t') = \int_0^L \frac{1}{2}\rho W h\, \omega^2 \hat{y}^2(x)\, dx$$

To get frequency:

$$K_{max} = W_{max}$$

$$\therefore \boxed{\omega = \sqrt{\frac{W_{max}}{\int_0^L \frac{1}{2}\rho W h\, \hat{y}^2(x)\, dx}}} \quad [\text{radians/s}]$$

ω: res. freq.

W_{max}: maximum potential energy

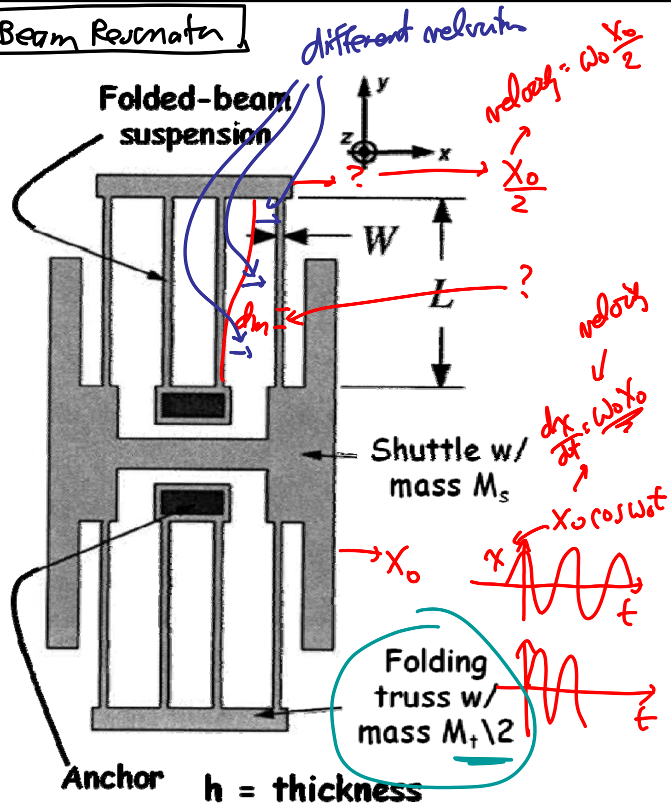
ρ: density of the structural material

W: beam width

h: " thickness

$\hat{y}(x)$: resonance mode shape

Folded Beam Resonator



- Derive an expression for the resonance frequency of the above structure

Use the Rayleigh-Ritz method:

$$KE_{max} = PE_{max}$$

Kinetic Energy: Find this!

$$KE_{max} = \underbrace{KE_s}_{\text{shuttle}} + \underbrace{KE_t}_{\text{truss}} + \underbrace{KE_b}_{\text{beams}}$$

$$= \frac{1}{2} v_s^2 M_s + \frac{1}{2} v_t^2 M_t + \frac{1}{2} \int v_b^2 \, dM_b$$

Velocity of Shuttle: $v_s = \omega_0 X_0$ ← maximum displacement of the shuttle

resonance freq.

$$\therefore KE_s = \frac{1}{2} v_s^2 M_s = \frac{1}{2} \omega_0^2 X_0^2 M_s$$

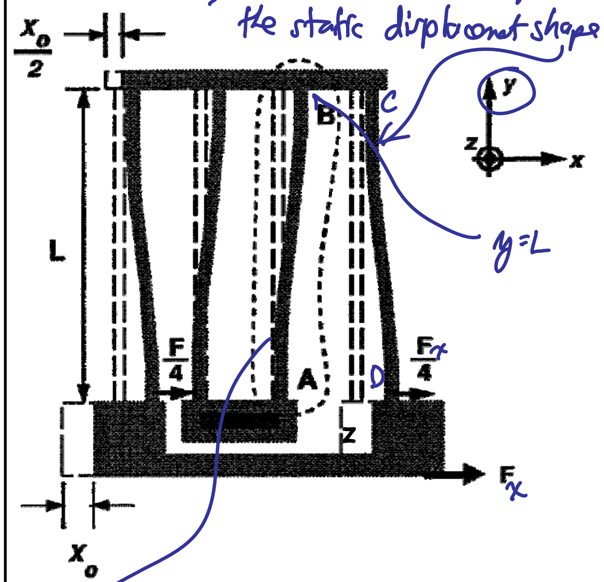
Velocity of Truss: $v_t = \frac{1}{2} v_s = \frac{1}{2} \omega_0 X_0$

mass of both trusses combined

$$\therefore KE_t = \frac{1}{2} \left(\frac{1}{2} \omega_0 X_0\right)^2 M_t = \frac{1}{8} \omega_0^2 X_0^2 M_t$$

Velocity of the beam segments:

⇒ assume the mode shape is the same as the static displacement shape



Segment AB:

$$\hat{x}(y) = \frac{F_x}{48 E I_z} \left(3Ly^2 - 2y^3\right) \qquad 0 \le y \le L \qquad (1)$$

At $y=L$: $x(L) = \frac{X_0}{2} = \frac{F_x L^3}{48EI_z}$ ← B.C.

Substitute into (1):

$$\hat{x}(y) = \frac{X_0}{2}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]$$

which yields for velocity

$$v_b(y)\Big|_{[AB]} = \frac{X_0}{2}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]\omega_0$$

Plugging into the expression for KE_b:

$$KE_{[AB]} = \frac{1}{2}\int_0^L \frac{X_0^2\omega_0^2}{4}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]^2 dM_{[AB]}$$

$$= \frac{X_0^2\omega_0^2 M_{[AB]}}{8L}\int_0^L \left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]^2 dy$$

Static Mass of beam [AB]

$$KE_{[AB]} = \frac{13}{280} X_0^2\omega_0^2 M_{[AB]}$$

For segment CD:

$$v_b(y)\Big|_{[CD]} = X_0\left[1 - \frac{3}{2}\left(\frac{y}{L}\right)^2 + \left(\frac{y}{L}\right)^3\right]\omega_0$$

Thus:

$$KE_{[CD]} = \frac{X_0^2\omega_0^2 M_{[CD]}}{2L}\int_0^L \left[1 - \frac{3}{2}\left(\frac{y}{L}\right)^2 + \left(\frac{y}{L}\right)^3\right]^2 dy$$

static mass of beam [CD]

$$KE_{[CD]} = \frac{83}{280} X_0^2\omega_0^2 M_{[CD]}$$

Let $M_b \triangleq$ total mass of all 8 beams.

Then: $M_{[AB]} = M_{[CD]} = \frac{1}{8}M_b$

Thus:

$$KE_b = 4KE_{[AB]} + 4KE_{[CD]} = \frac{6}{35}X_0^2\omega_0^2 M_b$$

and

$$KE_{max} = X_0^2\omega_0^2\left[\frac{1}{2}M_s + \frac{1}{8}M_t + \frac{6}{35}M_b\right]$$

PE_{max} → simply the work done to achieve max. deflection

$$PE_{max} = \frac{1}{2}k_x X_0^2$$

Thus, using Rayleigh-Ritz

$$KE_{max} = PE_{max}$$

$$\cancel{X_0^2}\,\omega_0^2\left[\frac{1}{2}M_s + \frac{1}{8}M_t + \frac{6}{35}M_b\right] = \frac{1}{2}k_x \cancel{X_0^2}$$

k_c

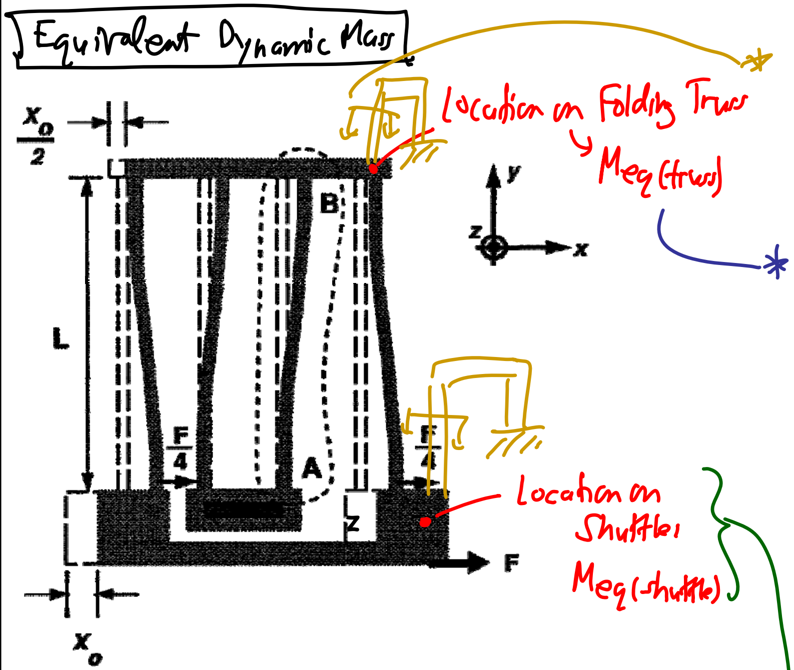
$$\boxed{\omega_0 = \left[\frac{k_x}{M_{eq}}\right]^{1/2} \quad \text{where } M_{eq} = M_s + \frac{1}{4}M_t + \frac{12}{35}M_b}$$

k_c

(Resonance Freq. of a Folded-Beam Suspended Shuttle)

- Go through Module 10 slides 21-31 on your own

Equivalent Dynamic Mass



Equivalent Mass:

$$\text{Equiv. Mass} = M_{eq\,x} = \frac{KE_{max}}{\frac{1}{2}V_x^2} = \frac{\frac{1}{2}\rho A \int_0^{\ell} v^2(x)\,dx}{\frac{1}{2}V_x^2}$$

$$M_{eq}(\text{shuttle}) = \frac{KE_{max}}{\frac{1}{2}V_{shuttle}^2} = \frac{\cancel{\omega_0^2 x_0^2 \left(\frac{1}{2}\right)}\left[M_s + \frac{1}{4}M_t + \frac{12}{35}M_b\right]}{\cancel{\frac{1}{2}\omega_0^2 x_0^2}}$$

$$M_{eq}(\text{shuttle}) = M_s + \frac{1}{4}M_t + \frac{12}{35}M_b$$

$$* \quad M_{eq}(\text{truss}) = \frac{\cancel{\omega_0^2 x_0^2 \left(\frac{1}{2}\right)}\left[M_s + \frac{1}{4}M_t + \frac{12}{35}M_b\right]}{\cancel{\frac{1}{2}}\left(\frac{1}{4}\right)\cancel{\omega_0^2 x_0^2}}$$

$$\therefore \quad M_{eq}(\text{truss}) = 4\left[M_s + \frac{1}{4}M_t + \frac{12}{35}M_b\right]$$

Equivalent Dynamic Stiffness

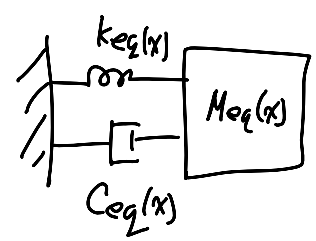
$$\omega_0 = \sqrt{\frac{k_{eq}(x)}{M_{eq}(x)}} \rightarrow \boxed{k_{eq}(x) = \omega_0^2 M_{eq}(x)}$$

⇒ large equiv. mass & stiffness go hand in hand!

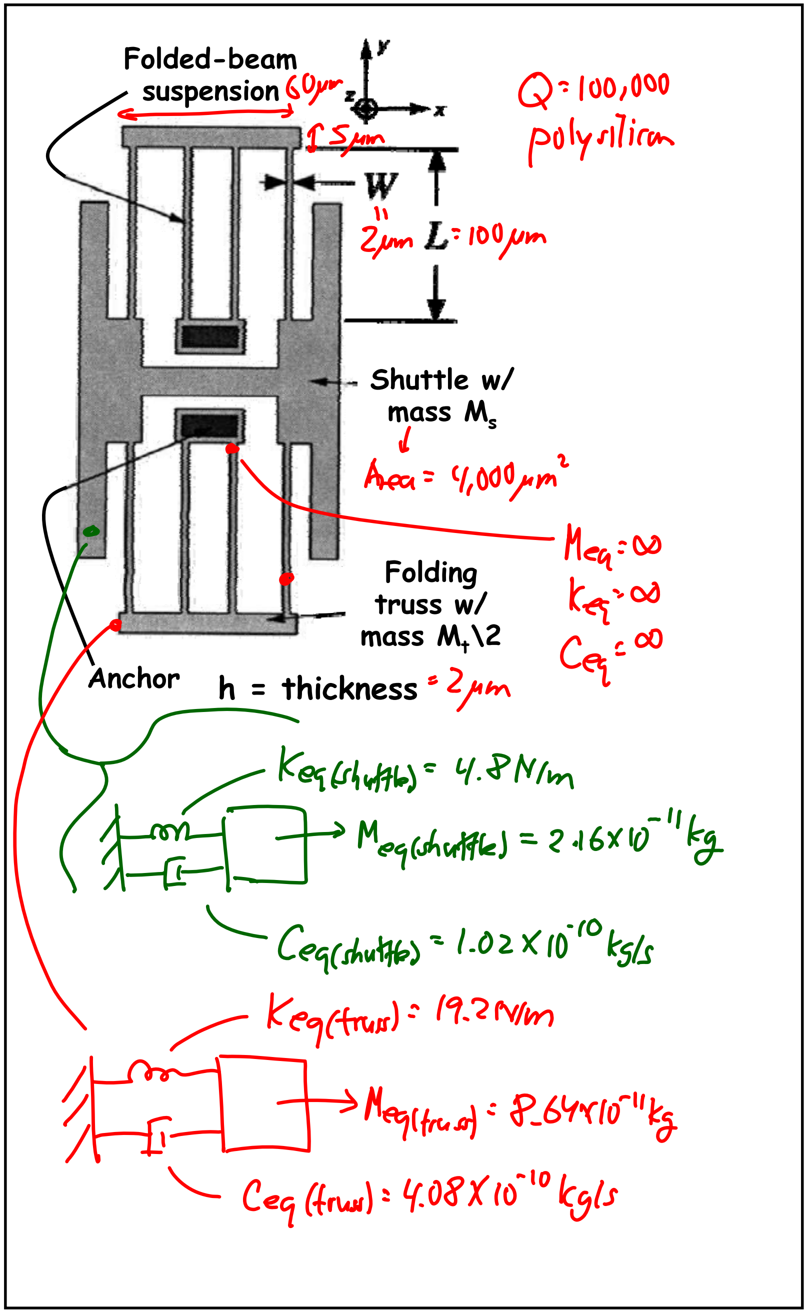
Equiv. Dynamic Damping

$$Q = \frac{\omega_0 M_{eq}(x)}{c_{eq}(x)} \begin{matrix} \sim L \\ \sim R \end{matrix} \rightarrow \boxed{c_{eq}(x) = \frac{\omega_0 M_{eq}(x)}{Q} = \frac{\sqrt{k_{eq}(x) M_{eq}(x)}}{Q}}$$

$c_{eq}(x)$ ← damping



} specified @ a single location x


Folded-beam suspension
60μm
5μm
y
z
x
Q = 100,000
polysilicon
W = 2μm
L = 100μm
Shuttle w/ mass M_s
Area = 4,000μm²
$M_{eq} = \infty$
$K_{eq} = \infty$
$C_{eq} = \infty$
Folding truss w/ mass $M_t\backslash 2$
Anchor
h = thickness = 2μm
$K_{eq(shuttle)} = 4.8 N/m$
$M_{eq(shuttle)} = 2.16 \times 10^{-11} kg$
$C_{eq(shuttle)} = 1.02 \times 10^{-10} kg/s$
$K_{eq(truss)} = 19.2 N/m$
$M_{eq(truss)} = 8.64 \times 10^{-11} kg$
$C_{eq(truss)} = 4.08 \times 10^{-10} kg/s$