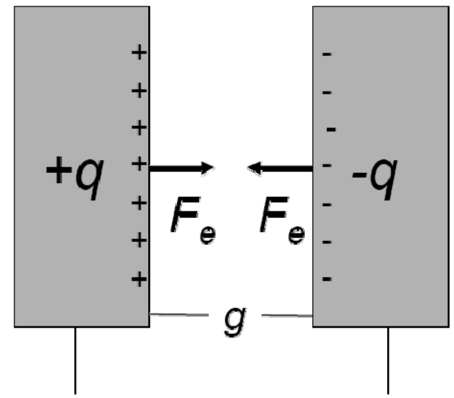
- **Announcements**:
- **Lecture schedule for the next 3 weeks:**

Monday	Tuesday	Wednesday	Thursday	Friday
9 Lecture 5-6:30 241 Cory	10 Lecture 3:30-5 Moffitt	11	15 Lecture 3:30-5 Moffitt	13
16 Discuss 3:30-5 241 Cory	17	18	19	20
23 Lecture 5-6:30 241 Cory	24 Lecture 3:30-5 Moffitt	25	26	27

- **Reading:** Senturia, Chpt. 5, Chpt. 6
- **Lecture Topics:**
  - Energy Conserving Transducers
    - Charge Control
    - Voltage Control
  - Parallel-Plate Capacitive Transducers
    - Linearizing Capacitive Actuators
    - Electrical Stiffness
  - Electrostatic Comb-Drive
    - 1st Order Analysis
    - 2nd Order Analysis

- **Last Time**:

$\boxed{\text{Charge-Control Case}}$



From $dW = Vdq + F_e dg$:

⇒ Force is given by:

$$F_e = \left.\frac{\partial W(q,g)}{\partial g}\right|_{q=const.} = \frac{\partial}{\partial g}\left(\frac{1}{2}\frac{q^2}{\epsilon A}g\right)$$

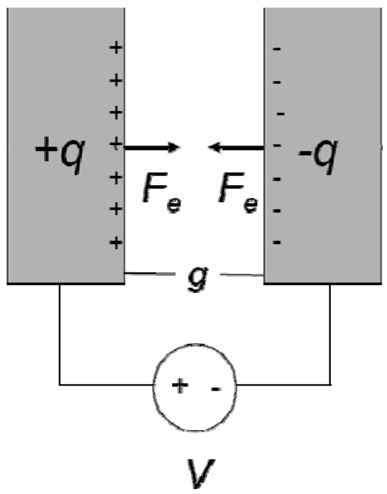
$\therefore \boxed{F_e = \frac{1}{2}\frac{q^2}{\epsilon A}}$ ⇒ independent of gap spacing!

⇒ Voltage is given by

$$V = \left.\frac{\partial W(q,g)}{\partial q}\right|_{g=const.} = \frac{\partial}{\partial q}\left(\frac{1}{2}\frac{q^2}{\epsilon A}g\right) = \frac{qg}{\epsilon A}$$

$$\boxed{\therefore V = \frac{q}{C}}$$ ✓

Voltage Control



Want to write

$F_e = f(V)$.

How can we do this?

Well, we have this:

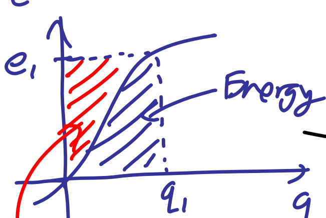
$dW = Vdq + F_e dg$

$\underbrace{W = W(q,g)}_{\text{can't work}}$

Need $W'(V,g)$.

To get this, do a Legendre transformation.

Effort (e.g., force, voltage, ...)



Displacement (e.g., displacement, charge, ...)

Energy $= W(q_1) = \int_0^{q_1} e\,dq = \int_0^{q_1} \Phi(q)\,dq$

Co-Energy:

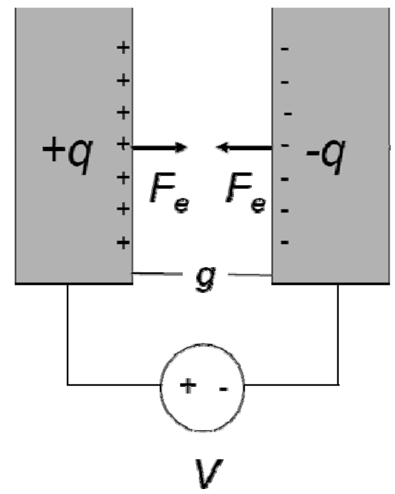
$W'(e_1) = \int_0^{e_1} q\,de = \int_0^{e_1} \Phi^{-1}(e)\,de$

⇒ Can define co-energy as:

$W'(e) = eq - W(q)$

For a linear system, these will be equal. *

Co-Energy Formulation



* → $W'(V,g) = qV - W(q,g)$

Differentially, this becomes:

$dW'(V,g) = (qdV + Vdq) - dW(q,g)$

[But: $dW(q,g) = F_e dg + Vdq$]

$\boxed{dW'(V,g) = qdV - F_e dg}$ ← Working co-energy expression!

Find co-energy in terms of voltage V:

$$W' = \int_0^V q(g,V')\,dV' = \int_0^V \left(\frac{\epsilon A}{g}\right)V'\,dV' = \frac{1}{2}\left(\frac{\epsilon A}{g}\right)V^2$$

$$= \frac{1}{2}CV^2$$

(as expected)

Electrostatic (a voltage-controlled) Force:

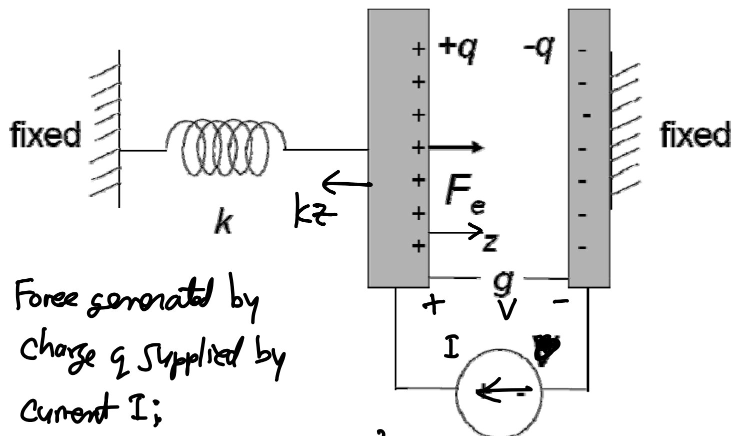
$$F_e = -\frac{\partial W'(V,g)}{\partial g}\Big|_{V=const} = -\frac{1}{2}\left(-\frac{\epsilon A}{g^2}\right)V^2 = \frac{1}{2}\frac{C}{g}V^2$$

depends on gap!

Charge:

$$q = \frac{\partial W'(V,g)}{\partial V}\Big|_{g=const} = \frac{\epsilon A}{g}V = CV \quad \text{(as expected)}$$

$\boxed{\text{Charge-Control of a Spring-Suspended C}}$



Force generated by charge q supplied by current I;

$$F_e = \frac{\partial W(q,g)}{\partial g}\Big|_q = \frac{q^2}{2\epsilon A}$$

Restoring force of spring: $F_{spring} = kz = F_e$

equilibrium

The gap:

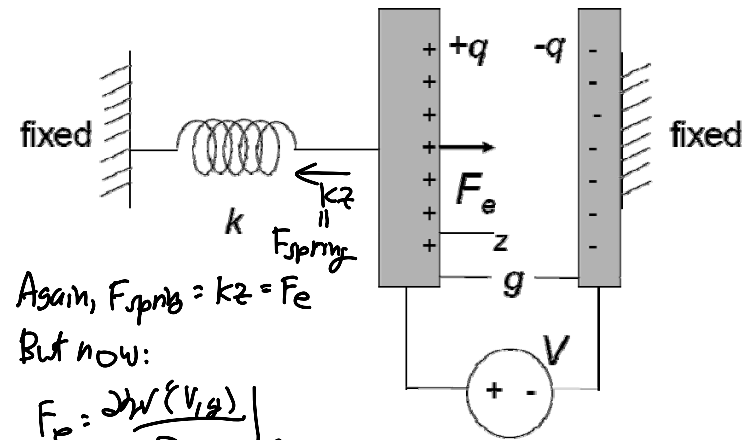
$$g = g_0 - z = g_0 - \frac{F_e}{k} = \boxed{g_0 - \frac{1}{2}\frac{q^2}{\epsilon A}\frac{1}{k} = g}$$

initial gap

↳ $q\uparrow$ can drive $g \to 0$

$$V = \frac{q}{C} = \frac{q}{\epsilon A}g = \boxed{\frac{q}{\epsilon A}\left(g_0 - \frac{1}{2}\frac{q^2}{\epsilon A}\frac{1}{k}\right) = V} \rightarrow V\downarrow \text{ as } g\downarrow$$

$\boxed{\text{Voltage-Control of a Spring-Suspended C}}$



Again, $F_{spring} = kz = F_e$

But now:

$$F_e = \frac{\partial W'(V,g)}{\partial g}\Big|_q$$

$$F_e = \frac{1}{2}\frac{\epsilon A}{g^2}V^2$$

And the gap:

$$g = g_0 - z = g_0 - \frac{F_e}{k} = \boxed{g_0 - \frac{1}{2}\frac{\epsilon A}{g^2}V^2 = g}$$

g shows up on both sides!

If $V\uparrow \rightarrow g\downarrow \rightarrow F_e\uparrow$

(+) Feedback!

If loop gain > 1, then this will go unstable!

Charge: (for a stable gap)

$$q = \frac{\partial W'(V,g)}{\partial V}\bigg|_g = CV \checkmark$$

$\boxed{\text{Stability Analysis}}$

⇒ determine under what conditions voltage-control will cause collapse of the plate...

$$F_{net} = F_e - F_{spring} = \underbrace{\frac{\epsilon A V^2}{2g^2}}_{F_e} - \underbrace{k(g_0 - g)}_{F_{spring}}$$

What happens when I change g by an increment dg?
↳ get an increment in net attractive force dF_{net}

$$\underset{(-)}{dF_{net}} = \frac{\partial F_{net}}{\partial g} dg = \left[-\frac{\epsilon A V^2}{g^3} + k\right] \underset{(-)}{dg}$$

If $g\downarrow \rightarrow dg = (-)$, then for stability need $F_{net}\downarrow \rightarrow dF_{net} = (-)$

This needs to be (+)! → otherwise the plate collapses

Thus:

$$\boxed{k > \frac{\epsilon A V^2}{g^3}}$$ (for a stable uncollapsed state)

$\boxed{\text{Pull-In Voltage \& Gap}}$

$V_{PI} \triangleq$ voltage @ which plates collapse

The plate goes unstable when:

$$k = \frac{\epsilon A V_{PI}^2}{g_{PI}^3} \quad (1)$$

$$F_{net} = 0 = \frac{\epsilon A V_{PI}^2}{2 g_{PI}^2} - k(g_0 - g_{PI}) \quad (2)$$

V_{PI} ← pull-in voltage

g_{PI} ← pull-in gap

Substituting (1) into (2):

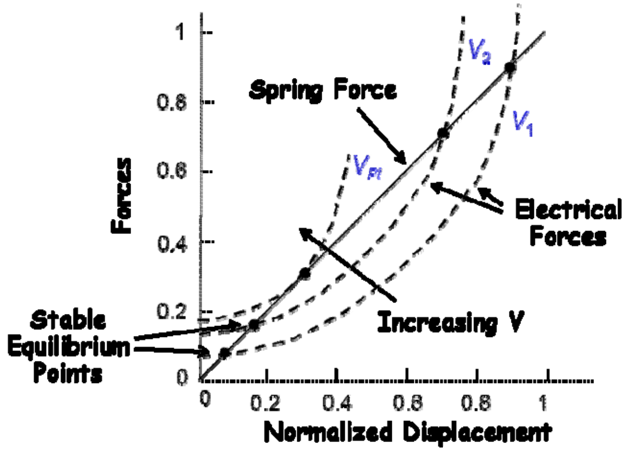
$$0 = \frac{\cancel{\epsilon A V_{PI}^2}}{2 \cancel{g_{PI}^2}} - \frac{\cancel{\epsilon A V_{PI}^2}}{g_{PI}^{\cancel{3}}}(g_0 - g_{PI})$$

$$\frac{g_0 - g_{PI}}{g_{PI}} = \frac{1}{2} \rightarrow g_0 = \frac{3}{2} g_{PI}$$

$$\therefore \boxed{g_{PI} = \frac{2}{3} g_0}$$

When the gap is driven by a voltage to (2/3) the initial spacing → collapse!

$$V_{PI} = \sqrt{\frac{k g_{PI}^3}{\epsilon A}} \rightarrow \boxed{V_{PI} = \sqrt{\frac{8}{27}\frac{k g_0^3}{\epsilon A}}}$$



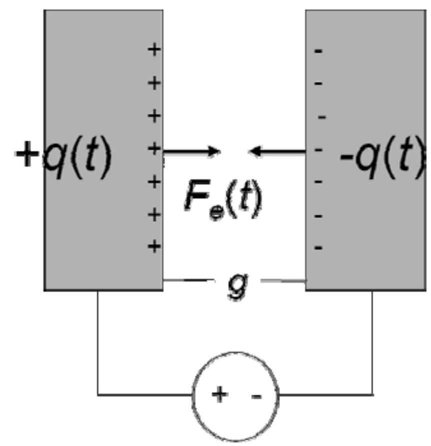
Advantages of Electrostatic Actuators:

- Easy to manufacture in micromachining processes, since conductors and air gaps are all that's needed → low cost!
- Energy conserving → only parasitic energy loss through I^2R losses in conductors and interconnects
- Variety of geometries available that allow tailoring of the relationships between voltage, force, and displacement
- Electrostatic forces can become very large when dimensions shrink → electrostatics scales well!
- Same capacitive structures can be used for both drive and sense of velocity or displacement
- Simplicity of transducer greatly reduces mechanical energy losses, allowing the highest Q's for resonant structures

Advantages of Electrostatic Actuators:

- Nonlinear voltage-to-force transfer function
- Relatively weak compared with other transducers (e.g., piezoelectric), but things get better as dimensions scale

$\boxed{\text{Linearizing the Voltage-to-Force Xfer Function}}$



$$v(t) = \underbrace{V_P}_{\text{Bias (dc)}} + \underbrace{v_i(t)}_{\text{Signal (ac)}}$$

$$F_e(t) = \frac{\partial W'}{\partial x} = \frac{\partial}{\partial x}\left[\frac{1}{2} C (v(t))^2\right]$$

$$= \frac{1}{2}\frac{\partial C}{\partial x}[v(t)]^2 = \frac{1}{2}\frac{\partial C}{\partial x}[V_P + v_i(t)]^2$$

$$= \frac{1}{2}\left[V_P^2 + 2V_P v_i(t) + \cancel{[v_i(t)]^2}\right]\frac{\partial C}{\partial x}$$

$$[V_P >> v_i(t)] \Rightarrow F_e(t) = \underbrace{\frac{1}{2}V_P^2\frac{\partial C}{\partial x}}_{\text{DC offset}} + \underbrace{V_P\frac{\partial C}{\partial x}v_i(t)}_{\text{AC drive signal}}$$

$$\left[C_o = \frac{\epsilon A}{g_o}\right]$$

$$C(x) = \frac{\epsilon A}{g_0 - x} = C_0\left(1 - \frac{x}{g_0}\right)^{-1} \approx C_0\left(1 + \frac{x}{g_0}\right)$$

$[x \ll g_0]$

$$\therefore \frac{\partial C}{\partial x} = \frac{C_0}{g_0}$$

$$\Rightarrow F_e(t) = \underbrace{\frac{1}{2}\frac{C_0}{g_0}V_P^2}_{\text{Can cancel by symmetry (later)}} + V_P \underbrace{\frac{C_0}{g_0}}_{\sim\text{const. for small amplitudes}} v_i(t)$$

linear!

This is small-signal analysis in electromechanical domain!