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**EE C245 - ME C218**  
**Introduction to MEMS Design**  
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**Prof. Clark T.-C. Nguyen**

Dept. of Electrical Engineering & Computer Sciences  
University of California at Berkeley  
Berkeley, CA 94720

**Lecture Module 13: Equivalent Circuits II**

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**Lecture Outline**

- Reading: Senturia, Chpt. 6, Chpt. 14
- Lecture Topics:
  - ↗ Input Modeling
    - Force-to-Velocity Equiv. Ckt.
    - Input Equivalent Ckt.
  - ↗ Current Modeling
    - Output Current Into Ground
    - Input Current
    - Complete Electrical-Port Equiv. Ckt.
  - ↗ Impedance & Transfer Functions

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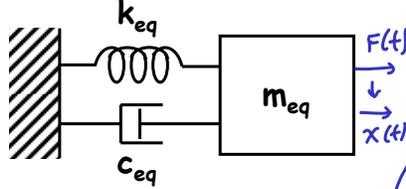


## Input Modeling

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## Electromechanical Analogies



$F(t) = F \cos(\omega t) \rightarrow x(t) = X \cos \omega t$

Equation of Motion:

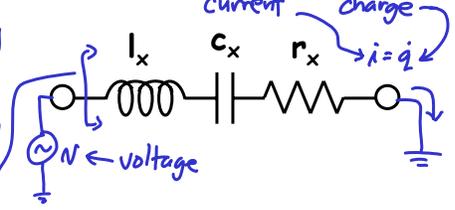
$$m_{eq} \ddot{x} + c_{eq} \dot{x} + k_{eq} x = F(t)$$

⇒ using phasor concepts:

$$F = j\omega m_{eq} \dot{x} + \frac{k_{eq}}{j\omega} x + c_{eq} \dot{x}$$

⇒ by analogy:

$F \rightarrow V$	$m_{eq} \rightarrow l_x$	$c_{eq} \rightarrow r_x$
$\dot{x} \rightarrow i$	$k_{eq} \rightarrow \frac{1}{C_x}$	



$v(t) = V \cos \omega t \rightarrow i(t) = I \cos \omega t$

Impedance looking in:

$$\frac{V}{I} = j\omega l_x + \frac{1}{j\omega C_x} + r_x$$

$$V = j\omega l_x i + \frac{(1/C_x)}{j\omega} i + r_x i$$

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### Bandpass Biquad Transfer Function

$F = j\omega m_{eq} \dot{x} + \frac{k_{eq}}{j\omega} x + C_{eq} \dot{x}$   
 $\Rightarrow$  Converting to full phasor form:  
 $F = (j\omega)(j\omega X) m_{eq} + \frac{k_{eq}}{j\omega} (j\omega X) + C_{eq} (j\omega X)$   
 $\frac{X}{F}(j\omega) = \frac{1}{k_{eq}} \left[ -\omega^2 \frac{m_{eq}}{k_{eq}} + 1 + j \frac{C_{eq}\omega}{k_{eq}} \right]^{-1} = \frac{1}{k_{eq}} \left[ -\left(\frac{\omega}{\omega_0}\right)^2 + 1 + j \frac{\omega}{Q\omega_0} \right]^{-1}$   
 $\left[ \frac{k_{eq}}{m_{eq}} = \omega_0^2, Q = \frac{m_{eq}\omega_0}{C_{eq}} = \frac{k_{eq}}{\omega_0 C_{eq}} \rightarrow \frac{k_{eq}}{C_{eq}} = Q\omega_0 \right]$

$$\frac{X}{F}(j\omega) = \frac{k_{eq}^{-1}}{1 - \left(\frac{\omega}{\omega_0}\right)^2 + j \frac{\omega}{Q\omega_0}}$$

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### Force-to-Velocity Relationship

- The relationship between input voltage  $v_1$  and force  $F_{d1}$ :
 
$$F_{d1} \approx -V_P \frac{\partial C_1}{\partial x} v_1$$
- When displacement  $x$  is the mechanical output variable:
 
$$\frac{X(s)}{F_{d1}(s)} = \frac{1}{k} \frac{\omega_0^2}{s^2 + (\omega_0/Q)s + \omega_0^2}$$
- When velocity  $v$  is the mechanical output variable:
 
$$\frac{v(s)}{F_{d1}(s)} = \frac{sX(s)}{F_{d1}(s)} = \frac{1}{k} \frac{\omega_0^2 s}{s^2 + (\omega_0/Q)s + \omega_0^2}$$

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**Force-to-Velocity Equiv. Ckt.**

• Combine the previous lumped LCR mechanical equivalent circuit with a circuit modeling the capacitive transducer → circuit model for voltage-to-velocity

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**Equiv. Circuit for a Linear Transducer**

• A transducer ...

- ↪ converts energy from one domain (e.g., electrical) to another (e.g., mechanical)
- ↪ has at least two ports
- ↪ is not generally linear, but is virtually linear when operated with small signals (i.e., small displacements)

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**Equiv. Circuit for a Linear Transducer**

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- For physical consistency, use a transformer equivalent circuit to model the energy conversion from the electrical domain to mechanical domain

$$\begin{bmatrix} e_2 \\ f_2 \end{bmatrix} = \begin{bmatrix} \eta & 0 \\ 0 & -\frac{1}{\eta} \end{bmatrix} \begin{bmatrix} e_1 \\ f_1 \end{bmatrix}$$

Describing Matrix

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**Electromechanical Equivalent Circuit**

- $e_2 = F_{d1}$ ,  $e_1 = v_1$ , just need  $\eta_1$ :
- From the matrix:  $e_2 = \eta e_1$

$$F_{d1} \approx -V_P \frac{\partial C_1}{\partial x} v_1 \rightarrow \eta_1 = \left| V_P \frac{\partial C_1}{\partial x} \right|$$

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## Output Modeling

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## Output Current Into Ground

- When the mass moves with time-dependent displacement  $x(t)$ , the electrode-to-mass capacitors  $C_1(x, t)$  and  $C_2(x, t)$  vary with time
- This generates an output current:

$$[q = CV] \Rightarrow i = \frac{dq}{dt} = C \frac{\partial V}{\partial t} + V \frac{\partial C}{\partial t}$$

$$i_2(t) = C_2(x, t) \frac{dV_2(t)}{dt} + V_2(t) \frac{dC_2(x, t)}{dt}$$

$$[V_2(t) = -V_P] \Rightarrow i_2 = -V_P \frac{dC_2}{dt} = -V_P \frac{\partial C_2}{\partial x} \frac{\partial x}{\partial t}$$

In phasor form:  $I_2(j\omega) = -V_P \frac{\partial C_2}{\partial x} (j\omega X)$

$$I_2(j\omega) = -j\omega V_P \frac{\partial C_2}{\partial x} X$$

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### Output Current Into Ground

$I_2(j\omega) = -j\omega V_p \frac{\partial C_2}{\partial x} X = -V_p \frac{\partial C_2}{\partial x} U$   
 $90^\circ$  phase lag  
 $(+)$   $(+)$   $\rightarrow I_2 = (-)$  when  $x = (+)$  ✓

- Again, model with a transformer:

Velocity  $\rightarrow U = \dot{x}$       Current  $\rightarrow I_2$

$f_2 = -\frac{1}{\eta_2} f_1 \rightarrow f_1 = -\eta_2 f_2$   
 $[f_1 = I_2, f_2 = U] \Rightarrow I_2 = -\eta_2 U$   
 $\therefore \eta_2 = \left| V_p \frac{\partial C_2}{\partial x} \right|$

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### Input Current Expression

Get  $I_1(j\omega)$ :

$$i_1(t) = C_1(x,t) \frac{dv_1(t)}{dt} + v_1(t) \frac{dC_1(x,t)}{dt}$$

$$[v_1(t) \cdot N_1 - V_p] \Rightarrow i_1 = C_1 \frac{dv_1}{dt} + [M - V_p] \frac{\partial C_1}{\partial x} \frac{\partial x}{\partial t}$$

$$\therefore I_1(j\omega) = C_1(j\omega V_1) + V_1 \frac{\partial C_1}{\partial x} (j\omega X) - V_p \frac{\partial C_1}{\partial x} (j\omega X)$$

$$= j\omega C_1 V_1 + j\omega V_1 \frac{\partial C_1}{\partial x} X - j\omega V_p \frac{\partial C_1}{\partial x} X$$

$[V_1 \ll V_p] \Rightarrow I_1(j\omega) = \underbrace{j\omega C_1 V_1}_{\text{Feed-through Current}} - \underbrace{j\omega V_p \frac{\partial C_1}{\partial x} X}_{\text{Motional Current (due to mass motion)}}$

@ DC:  $x = \frac{F_{dl}}{k} = -\frac{1}{k} V_p \left( \frac{\partial C_1}{\partial x} \right) V_1$   
 @ resonance:  $x = \frac{Q F_{dl}}{jk} = -\frac{Q}{jk} V_p \frac{\partial C_1}{\partial x} V_1$   
 $\hookrightarrow 90^\circ$  phase lag

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### Input Current Expression (cont)

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Thus: (@resname)  

$$I_1(j\omega) = j\omega C_1 V_1 + j\omega_0 \left( V_P \frac{\partial C_1}{\partial x} \right)^2 \frac{Q}{jk} V_1$$

$$= j\omega_0 C_1 V_1 + \omega_0 \frac{Q}{k} \eta^2 V_1$$

90° phase-shifted from  $V_1$  → This is a Capacitor in shunt w/ the input.  
 In phase w/  $V_1$  → This is an effective resistance seen looking into Electrode 1

Motional Resistance:  

$$R_{x1} \frac{V_1}{I_1} = \frac{k}{\omega_0 Q \eta^2} = \frac{m \omega_0}{Q \eta^2} = \frac{b}{\eta^2} = R_{x1}$$

(The equivalent ckt. better get this right!)

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### Complete Electrical-Port Equiv. Circuit

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Static electrode-to-mass overlap capacitance

$l_x = m$   
 $c_x = \frac{1}{k}$   
 $r_x = b$

$\eta_{e1} = V_P \frac{\partial C_1}{\partial x} = V_P \frac{C_{o1}}{d_1}$   
 $\eta_{e2} = V_P \frac{\partial C_2}{\partial x} = V_P \frac{C_{o2}}{d_2}$

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### Input Impedance Into Port 1

• What is the impedance seen looking into port 1 with port 2 shorted to ground?

From our transformer model:  $\begin{bmatrix} e_2 \\ f_2 \end{bmatrix} = \begin{bmatrix} n & 0 \\ 0 & -\frac{1}{n} \end{bmatrix} \begin{bmatrix} e_1 \\ f_1 \end{bmatrix} \Rightarrow e_2 = ne_1 \rightarrow e_1 = \frac{e_2}{n}$   
 $f_2 = -\frac{1}{n} f_1 \rightarrow f_1 = -n f_2$

$\frac{e_1}{f_1} = \frac{e_2}{n} \left( \frac{1}{-n f_2} \right) = -\frac{1}{n^2} \frac{e_2}{f_2} \rightarrow \frac{N_1}{i_1} = Z_i = -\frac{1}{n^2} \frac{F_2}{f_2} = \frac{1}{n^2} Z_x$

$Z_i = \frac{1}{n^2} \left( j\omega L_x + \frac{1}{j\omega C_x} + r_x \right) = j\omega \left( \frac{L_x}{n^2} \right) + \frac{1}{j\omega \left( n^2 C_x \right)} + \frac{r_x}{n^2}$

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### Input Impedance Into Port 2

• What is the impedance seen looking into port 2 with port 1 shorted to ground?

$\frac{N_1}{i_2} = Z_i = \frac{1}{n^2} \left( j\omega L_x + \frac{1}{j\omega C_x} + r_x \right) = j\omega \left( \frac{L_x}{n^2} \right) + \frac{1}{j\omega \left( n^2 C_x \right)} + \frac{r_x}{n^2}$

Note: These are not the same as  $L_{x1}$ ,  $C_{x1}$ , &  $R_{x1}$ !

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### Port 1 to 2 TransG Across the Circuit

• What is the transconductance from port 1 to port 2 with port 2 shorted to ground?

$$\dot{x} = \frac{1}{\eta_{e1}} \dot{i}_i$$

$$\dot{i}_o = \eta_{e2} \dot{x} \rightarrow \dot{i}_o = \frac{\eta_{e2}}{\eta_{e1}} \dot{i}_i = \frac{\eta_{e2}}{\eta_{e1}} \left( \frac{N_1}{Z_x} \right) \dot{i}_i = \frac{\eta_{e2}}{\eta_{e1}} N_1 \left[ \eta_{e1}^2 \frac{1}{j\omega l_x + \frac{1}{j\omega c_x} + r_x} \right]$$

$$\therefore \frac{\dot{i}_o}{N_1} (j\omega) = \frac{\eta_{e1}\eta_{e2}}{j\omega l_x + \frac{1}{j\omega c_x} + r_x} = \left[ j\omega L_{x12} + \frac{1}{j\omega C_{x12}} + R_{x12} \right]^{-1} \begin{cases} L_{x12} = \frac{l_x}{\eta_{e1}\eta_{e2}} \\ C_{x12} = \eta_{e1}\eta_{e2}c_x \\ R_{x12} = \frac{r_x}{\eta_{e1}\eta_{e2}} \end{cases}$$

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### Port 1 to 2 $v_i$ -to- $i_o$ Transfer Function

$$\frac{\dot{i}_o}{N_1} (j\omega) = \frac{\eta_{e1}\eta_{e2}}{j\omega l_x + \frac{1}{j\omega c_x} + r_x} = \left[ j\omega L_{x12} + \frac{1}{j\omega C_{x12}} + R_{x12} \right]^{-1} \begin{cases} L_{x12} = \frac{l_x}{\eta_{e1}\eta_{e2}} \\ C_{x12} = \eta_{e1}\eta_{e2}c_x \\ R_{x12} = \frac{r_x}{\eta_{e1}\eta_{e2}} \end{cases}$$

Separate freq. response & magnitude:

$$\frac{\dot{i}_o}{N_1} (s) = \frac{1}{sL_x + \frac{1}{sC_x} + R_x} = \frac{sC_x}{s^2L_xC_x + 1 + sC_xR_x} = \frac{s(\frac{1}{L_x})}{s^2 + \frac{1}{L_xC_x} + s(\frac{R_x}{L_x})}$$

$$\left[ \frac{1}{L_xC_x} = \omega_0^2, Q = \frac{\omega_0 L_x}{R_x} \rightarrow \frac{R_x}{L_x} = \frac{\omega_0}{Q} \right] \Rightarrow \frac{\dot{i}_o}{N_1} (s) = \frac{1}{R_x} \frac{s(\omega_0/Q)}{s^2 + s(\omega_0/Q) + \omega_0^2} = \frac{1}{R_x} \mathcal{H}(s)$$

$$\mathcal{H}(s) = \frac{s(\omega_0/Q)}{s^2 + s(\omega_0/Q) + \omega_0^2}$$

$s=0: \mathcal{H}(0) = 0$   
 $s=j\omega_0: \mathcal{H}(j\omega_0) = 1$   
 $s=\infty: \mathcal{H}(\infty) = 0$

This will always be the same. Thus, could just work @ resonance & just multiply by  $\mathcal{H}(s)$ .

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### Condensed Equiv. Circuit (Symmetrical)

If  $\eta_{e1} = \eta_{e2}$ , then ...

Holds for the symmetrical case, where port 1 and port 2 are identical

where

$$L_x = \frac{m}{\eta_e^2}$$

$$C_x = \frac{\eta_e^2}{k}$$

$$R_x = \frac{b}{\eta_e^2}$$

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### Phasings of Signals

• Below: plots of resonance electrical and mechanical signals vs. time, showing the phasings between them

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